General Relativity and Modified Gravity

Dimitrios Germanis

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Abstract

In this paper we present a detailed review of the most widely accepted theory of gravity: general relativity. We review the Einstein-Hilbert action, the Einstein field equations, and we discuss the various astrophysical tests that have been performed in order to test Einstein's theory of gravity. We continue by looking at alternative formulations, such as the Palatini formalism, the Metric-Affine gravity, the Vierbein formalism, and others. We then present and analytically discuss a modification of General Relativity via the Chern-Simons gravity correction term. We formulate Chern-Simons modified gravity, and we provide a derivation of the modified field equations by embedding the 3D-CS theory into the 4D-GR. We continue by looking at the applications of the modified theory to CMB polarization, and review the various astrophysical tests that are used to test this theory. Finally, we look at f(R) theories of gravity and specifically, f(R) in the metric formalism, f(R) in the Palatini formalism, f(R) in the metric-affine formalism, and the various implications of these theories in cosmology, astrophysics, and particle physics.

Contents

1 Introduction

Despite the fact that gravity is the fundamental interaction which is so much related to our everyday experience, it still remains the most mysterious and enigmatic interaction from all the others. The gravitational force is the one most easily conceived of, without any deep and sophisticated knowledge, and was the first one to be tested experimentally due to the nature and simplicity of the experiments conducted and the apparatus used[41].

Galileo Galilei was the first to introduce the pendulum and inclined planes to the study of terrestrial gravity at the end of the 16th century. Gravity played an important role in the development of Galileo's ideas about the necessity of experiment in the study of Science, which had a great impact on modern scientific thinking. However, it was not until 1665, when Isaac Newton introduced the now renowned inverse-square gravitational force law, that terrestrial gravity was actually related to celestial gravity in a single theory. Newton's theory made correct predictions for a variety of phenomena at different scales, including both terrestrial experiments and planetary motion.Newton's contribution to gravity, quite apart from his enormous contribution to physics overall, is not restricted to the expression of the inverse square law. Much attention should be paid to the conceptual basis of his gravitational theory, which incorporates two key ideas[41],[113]:

- The idea of absolute space, i.e. the view of space as a fixed, unaffected structure; a rigid arena where physical phenomena take place.
- The idea of what was later called the Weak Equivalence Principle which, expressed in the language of Newtonian theory, states that the inertial and the gravitational mass coincide.

In 1855, Urbain Le Verrier observed a 35 arc-second excess precession of Mercurys orbit and later on, in 1882, Simon Newcomb measured this precession more accurately to be 43 arc-seconds. This experimental fact was not predicted by Newton's theory. It should be noted that Le Verrier initially tried to explain the precession within the context of Newtonian gravity, attributing it to the existence of another, yet unobserved, planet whose orbit lies within that of Mercury. He was apparently influenced by the fact that examining the distortion of the planetary orbit of Uranus in 1846 had led him, and, independently, John Couch Adams, to the discovery of Neptune and the accurate prediction of its position and momenta. However, this innermost planet was never found[41],[113].

However in 1893, Ernst Mach stated what was later called by Albert Einstein's: Machs principle. This is the first constructive attack to Newton's idea of absolute space after the 18th century debate between Gottfried Wilhelm von Leibniz and Samuel Clarke (Clarke was acting as Newtons spokesman) on the same subject, known as the LeibnizClarke Correspondence. Machs idea can be considered as rather vague in its initial formulation and it was essentially brought into the mainstream of physics later on by Einstein along the following lines: Inertia originates in a kind of interaction between bodies. This is obviously in contradiction with Newton's ideas, according to which inertia was always relative to the absolute frame of space. However, there was another more clear interpretation given by Dicke: The gravitational constant should be a function of the mass distribution in the Universe. This is different from Newton's idea of the gravitational constant as being universal and unchanging. Now Newton's basic axioms have to be reconsidered[41],[113].

Newton's theory was very successful in explaining the various aspects of gravity at that time. Newton's theory is also a classical theory, and has successfully described the physical world around us therefore it can be considered as a very consistent theory, although not necessarily the most right one. The question is how consistent a theory is rather how 'right' it is. The theory was able to explain within a couple of years of its formulation all questions posed at that time[113].

But it was not until 1905, when Albert Einstein completed Special Relativity, that Newtonian gravity would have to face a serious challenge. Einstein's new theory, which managed to explain a series of phenomena related to non-gravitational physics, appeared to be incompatible with Newtonian gravity. Relative motion and all the linked concepts had gone well beyond Galileo and Newton ideas and it seemed that Special Relativity should somehow be generalised to include non-inertial frames. In 1907, Einstein introduced the equivalence between gravitation and inertia and successfully used it to predict the gravitational red-shift. Finally, in 1915, he completed the theory of General Relativity, a generalisation of Special Relativity which included gravity and any accelerated frame. The theory matched perfectly the experimental result for the precession of Mercury's orbit, as well as other experimental findings like the Lense-Thirring gravitomagnetic precession (1918) and the gravitational deflection of light by the Sun, as measured in 1919 during a Solar eclipse by Arthur Eddington. GR overthrew Newtonian gravity and continues to be up to now an extremely successful and well-accepted theory for gravitational phenomena. As mentioned before, and as often happens with physical theories, Newtonian gravity did not lose its appeal to scientists. It was realised, of course, that it is of limited validity compared to

GR, but it is still sufficient for most applications related to gravity. What is more, in weak field limit of gravitational field strength and velocities, GR inevitably reduces to Newtonian gravity. Newtons equations for gravity might have been generalised and some of the axioms of his theory may have been abandoned, like the notion of an absolute frame, but some of the cornerstones of his theory still exist in the foundations of GR, the most prominent example being the Equivalence Principle, in a more suitable formulation of course[41],[113].

General Relativity together with quantum field theory are considered to be the backbones of modern physics. The theory is given in the language of differential geometry and was the first such mathematical physics theory, leading the way for other mathematical theories in physics such as the gauge theories and string theories. One of the most astonishing facts about GR is that almost after an entire century it hasn't changed at all. How spacetime behaves on macroscopic scales is best described by the Einstein's Field Equations[1]:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor and G is the Newton's constant of gravitation. It is precisely these equations that are thought to govern the expansion of the Universe, the behavior of black holes, the propagation of gravitational waves, and the formation of all structures in the Universe, from planets to stars, to galaxies and clusters/superclusters of galaxies. However, in the microscopic scales GR is not an adequate theory[1].

Even though General Relativity is a very successful theory this didn't stop alternatives being proposed. Even a little after the publication of the theory by Einstein, proposals were made in order to extend the theory, and incorporate it in a larger, more unified theory. Examples of this are the Eddington's theory of connections, Weyl's scale independent theory, and the higher dimensional theories of Kaluza and Klein[1]. There are many more proposed since then and there are several modification of GR that the reader can find in a very extended and detail review, Modified Gravity and Cosmology[1]. To mention a few, alternative theories of gravity with extra fields such as scalar-tensor theories(Brans-Dicke Theory), Einstein–Ether Theories, Bimetric Theories. We can also find higher derivative theories of gravity such as Horava-Lifschitz gravity, and Galileons[1].

Before presenting the theory of GR let us define what the theory actually means. Depending on the point of view, for cosmology GR is just the set of 10-non-linear partial PDE's, called the Einstein Fields equations. For particle physics, it refers to any dynamical theory of spin-2 fields that incorporates general covariance as general relativity, even if the field equations are different. In other words the GR refers to the theory that simultaneously exhibits general covariance, universal couplings to all matter fields, and satisfies Einstein's Field equations. Therefore any deviation from these principles is what we call modified gravity. However all proposals of modified theories respect general covariance as well as the universality of free fall.[1],[2],[33].[41],[113].

There are, however, some ambiguities involved. For example the 'matter fields' can be subjective. This is very true when we deal with exotic matter which can be used to explain the apparent late-time acceleration of the Universe. Furthermore, the Einstein's Field equations are well-known in four-dimensions. But what if we include more dimensions and then we may choose to either derive these field equations from an Einstein-Hilbert action in the higher dimensional space-time, or to the effective set of equations in four-dimensions. The above two possible definitions are not consistent and moreover we don't know whether in the EFE's we have include a cosmological constant or not. If not, then we can claim that this theory is a modified theory of gravity[1],[2],[33].

It is beyond the scope of this review to present the wide variety of modifications of GR. In this paper, however, we will present a detailed review of GR, by starting with a discussion of the main principles the theory is based on, and continuing with the Einstein-Hilbert action and the Einstein Field equations. We will also see the various astrophysical tests that have been performed so far to validate the theory. Next, we will see a very interesting modification of GR, namely Chern-Simons modified gravity, We will present the modified action, derive the modified field equations, discuss parity violation in the CS theory, and the application to CMB polarization. We will also examine the two very important consequences of parity violation, namely, cosmological and gravitational birefringence, and finally we will present the astrophysical tests for this theory. In the last chapter, we will see the very-well known f(R) theories-or actions that are a function of the Ricci scalar, either linear or non-linear. Since the literature for f(R) theories of gravity is extremely large, I have decided to present the very basics, such as a few toy models, and the f(R) in the Metric, Palatini, and Metric-Affine formalisms.

2 General Relativity

2.1 Foundations of GR

Requirements for Validity

In order to construct a relativistic theory of gravity it is of primarily importance to establish the properties it must satisfy in order for it to be considered viable. These include foundation requirements, such as the Universality of free fall and the isotropy of space,as well as compatibility with a variety of different observations involving the propagation of light and the orbits of massive bodies. In this section we will discuss the gravitational experiments and observations that have so far been performed in these environments, and what they tell us about the theory of relativity and the principles that a theory must obey in order for it to stand a chance of being considered viable[1].

The ABC of General Relativity

There are five principles [2] which, explicitly or explicitly, guided Einstein in his search. They are:

1) Mach's Principle
2) The Principle of Equivalence
3) The Principle of Covariance
4) The Principle of Minimal Gravitational Coupling
5) The Correspondence Principle

Mach's Principle

Mach's Principle was proposed by Mach in 1893, and the starting point of which is that there is no meaning to the concept of motion, but only to that of relative motion. For example, a body in an otherwise empty Universe cannot be said to be in motion according to Mach, since there is nothing to which the body's motion can be referred. Moreover, in a populated Universe, it is the interaction between all matter in the Universe which is the source of all inertial effects. In our Universe, the bulk of the matter resides in what is called the 'fixed stars'. Then from Mach's viewpoint, an inertial frame is a frame in some privileged state of motion relative to the average motion of fixed stars. Hence, it is the fixed stars through their masses, distribution, and motion which determine a local inertial frame[2].

Mach's Principle can be incorporated in the following [2] three statements:

M1. The matter distribution determines the geometry
M2. If there is no matter then there is no geometry
M3. A body in an otherwise empty Universe should possess no inertial properties

The Principle of Equivalence

Before we consider the Principle of Equivalence in GR let us consider the Principle of Equivalence in the Newtonian Theory of Gravity. According to this principle[1],[2],[3],[4],[5],[6] in the Newtonian theory:

All bodies in a given gravitational field will move in the same manner, if initial conditions are the same. In other words, in a given gravitational field, all bodies move with the same acceleration. In the absence of a gravitational field, all bodies move with the same acceleration relative a given non-inertial frame of reference. Therefore the Principle of Equivalence in the Newtonian Theory states that: locally any non-inertial frame of reference is equivalent to a certain gravitational field.

Globally, 'actual' gravitational fields can be distinguished from corresponding non-inertial frames of reference by their behavior at infinity: Gravitational fields generated by gravitational objects decay with distance. In Newton's theory the motion of a test particle is determined by the following [1],[2],[3],[4],[5],[6] equation of motion:

$$m_{in}a = -m_{gr}\nabla\phi \tag{1}$$

where a is the acceleration of the test particle, ϕ is the Newtonian Potential of the gravitational field,m_{in} is the inertial mass of the test particle, m_{gr} is the gravitational mass of the test particle, which is the gravitational analogue of the electric charge in the theory of Electromagnetism. The fundamental property of gravitational field, that all test particles move with the same acceleration for a given potential ϕ,is explained within the frame of

Newtonian Theory just by the following[1],[2],[3],[4],[5],[6] 'coincidence.

$$\frac{m_{in}}{m_{gr}} = 1 \tag{2}$$

Above we introduced the idea of the test particle or otherwise a particle that has negligible mass in comparison with massive gravitating body that creates the gravitational field. Hence we define a gravitational test particle, to be a test particle which experiences a gravitational field but does not itself alter the field or contribute to the field.

The Principle of Equivalence has three different forms in GR, however in general it can be expressed in the following way[6]: A uniform gravitational field is equivalent to, which means is not distinguishable from, uniform acceleration. In practise this means that a person cannot feel locally the difference between the standing on the surface of a gravitating object and moving away in a rocket with the same acceleration. According to Einstein these effects are actually the same.

The important consequence of the equivalence is that any gravitational field can be eliminated in the free-falling frames of reference, which are called local inertial frames or local Galilean frames. In other words there is no experiment to distinguish between being weightless far away from gravitational objects in space and being in free-fall in a gravitational field[6].

As mentioned above the Principle of Equivalence can be defined also in the following three ways according to certain conditions that need to be satisfied[1]:

• The Weak Equivalence Principle(WEP): All uncharged,freely falling test particles follow the same trajectories, once an initial position and velocity have been prescribed.
• Einstein's Equivalence Principle(EEP): The WEP is valid, and furthermore in all freely falling frames one recovers (locally and up to tidal gravitational forces)the same laws of special relativity, independent of position and velocity.
• The Strong Equivalence Principle(SEP): The WEP is valid for massive gravitating objects as well as test particles, and in all freely falling frames one recovers (locally and up to tidal gravitational forces) the same special relativistic physics, independent of position and velocity.

Furthermore let us consider the equivalence principles and we will not assume instantly that any of these principles are valid, but will rather reflect on what can be said about them experimentally[1]. This will allows us to separate out observations that test the equivalence principles, from observations that test the different gravitational theories that obey these principles, and this is an approach pioneered by Dicke[9].

The least stringent of the equivalence principles is the WEP. The best evidence in support of the WEP still comes from Eotvos type experiments that use a torsion balance to determine the relative acceleration of two different materials toward distant astronomical bodies. In reality these materials are self-gravitating, but their mass is usually small enough enough that they can effectively be considered to be non-gravitating test particles in the gravitational field of the astrophysical body[1].

Using beryllium and titanium the tightest constraint on the relative difference in accelerations of the two bodies, a_1 and a_2, is currently[1],[10]

$$\eta = 2|\frac{a_1 - a_2}{a_1 + a_2}| = (0.3 \pm 1.8) \times 10^{-13} \quad (3)$$

This is an improvement of around 4 orders of magnitude of the original results of Eotvos from 1922 [11]. It is expected that this can be improved upon by up to a further 5 orders of magnitude when space based tests of the equivalence principle are performed[1],[12]. These null results are generally considered to be a very tight constraint on the foundations of any relativistic gravitational theory if it is to be thought of as viable, that is, the WEP must be satisfied, at least up to the accuracy specified in the above equation.

let us now consider the gravitational red-shift of light. This is one of the classic tests of General Relativity, suggested by Einstein himself in 1926[1],[13]. If we accept energy momentum conservation in a closed system then it is only really a test of the WEP, and is superseded in its accuracy by the Eotvos experiment we have just discussed. The argument for this is the following[1],[9],[14]:

Consider an atom that initially has an inertial mass M_i and gravitational mass M_g. The atom starts near the ceiling of a lab of height h, in a static gravitational field of strength g, and with an energy reservoir on the lab floor beneath it. The atom emits a photon of energy E that then travels down to the lab floor, such that its energy is blue-sifted by the gravitational field to E' when it is collected by the reservoir. This process changes the the inertial and gravitational masses of the atom to M_i' and M_g', respectively. The atom is then lowered to the floor, a process which lowers its total energy to $M_g gh$. At this point, the atom re-absorbs a photon from the reservoir with energy $E' = (M_i'' - M_i')c^2$ and is then raised to its initial position of the ceiling. This last process raises its energy by $M_g'' gh$, where M_i'' and M_g'' are the inertial and gravitational masses of the atom after re-absorbing the photon. The work done in lowering and raising the atom in this way is given by $w = (M_g'' - M_g')gh$. Recalling that the energy gained by the photon in travelling from the lab ceiling to the lab floor is $E' - E$. From the principle of conservation of energy we have that $E' - E = w = (M_g'' - M_g')gh$. Now, if the WEP is obeyed then $M_i = M_g$, and the above equation becomes $E' - E = E'gh$. This is nothing more than the usual expression for the gravitational red-shift. Crucial here is the assumption that local position invariance is valid so both M_i and M_g are independent of where they are in the lab.

If the laws of physics are position independent,and energy is conserved, gravitational red-shift then simply tests the equivalence of gravitational and inertial masses, which is what the Eotvos experiment does to higher accuracy. Alternatively, if we make the WEP to be tightly constrained by the Eotvos experiment, then gravitational red-shift experiments can be used to gain high precision constraints of the laws of Physics[1],[15]. The gravitational red-shift effect by itself, however, does not appear to be able to distinguish between the different theories that obey the WEP and local position invariance. In Dicke's approach[9] it should therefore be considered as a test of the foundations of relativistic gravitational theories, rather than a test of the theories themselves[1].

The most stringent equivalence principle is the EEP[1]. Testing this, is a considerable more demanding task than was the case for the WEP, as one now not only has to show that different particles follow the same trajectories, but also that a whole set of relativistic laws are valid in the rest frames of these particles. Despite the difficulties involved with this, there is still compelling evidence that the EEP should also be considered valid to high accuracy[1]. The most accurate and direct of this evidence is due to the Hughes-Drever experiments[16],[17], which test the local spatial anisotropies by carefully observing the shape and spacing of atomic spectral lines. The basic idea here is to determine if any gravitational fields beyond a single rank-2 tensor are allowed to couple directly to matter fields. To see why this is of importance, let us first consider a number of point-like particles coupled to a single rank-2 tensor $g_{\mu\nu}$. The Lagrangian density for such a set of particles is given[1] by:

$$\mathcal{L} = \sum \int m_I \sqrt{-g_{\mu\nu} u^\mu u^\nu} d\lambda \tag{4}$$

where m_I are the masses of the particles, and u^μ is their 4-velocity measured with respect to some parameter λ. Using the variational principle we derive the Euler-Lagrange equations which tell us that the particles in the above equation follow geodesics of the metric $g_{\mu\nu}$, and Riemannian geometry tells us that at any point we can choose coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$ locally. We therefore recover Special Relativity at every point[1], and the EEP is valid.

Consider the case where the matter fields couple to two rank-2 tensors then the above argument falls apart. In this case the corresponding Lagrangian density of the two particles is[1]:

$$\mathcal{L} = \sum \int [m_I \sqrt{-g_{\mu\nu} u^\mu u^\nu} + n_I \sqrt{-h_{\mu\nu} u^\mu u^\nu}] d\lambda \tag{5}$$

where $h_{\mu\nu}$ is the new metric tensor, and n_I is the coupling of each particle to that field. The particles above can now no longer be thought of as following geodesics of any one metric as the new Euler-Lagrange equations[1]are not in the form of geodesic equations. Hence we don't have Riemannian Geometry here which we can use to locally transform to the Minskowski space-time and so the EEP is violated. The relevance of this discussion for the Hughes-Drever experiments is that EEP violating couplings, such that the ones just described above, cause the types of anisotropies that these experiments constrain. In this case the 4-momentum of the test particle is given[1]by:

$$p_\mu = \frac{m g_{\mu\nu} u^\nu}{\sqrt{-g_{\alpha\beta} u^\alpha u^\beta}} + \frac{n h_{\mu\nu} u^\nu}{\sqrt{-h_{\alpha\beta} u^\alpha u^\beta}} \tag{6}$$

and as $g_{\mu\nu}$ and $h_{\mu\nu}$ cannot in general be made to be simultaneously spatially isotropic, we then have that p_μ is spatially isotropic, and should cause the type of shifts and broadening of spectral lines that Hughes-Drever type experiments are designed to detect[1]. The current tightest constrains are around 5 orders of magnitude tighter than the original experiments of Hughes-Drever[18],[19], and yields constraints of the order:

$$n \leq 10^{-27} m \tag{7}$$

so that couplings to the second metric must be very weak in order to be observationally viable. This result strongly supports the argument that matter fields must be coupled to a single rank-2 tensor only. It then follows that particles follow geodesics of this metric, that we can recover Special Relativity at any point and hence that the EEP is valid[1].

Beyond direct experimental tests such as the Hughes-Drever-type experiments, there are also theoretical reasons to think that the EEP is valid to high accuracy. This is a conjecture attributed to Schiff, that states: 'Any complete and self-consistent gravitational theory that obeys the WEP must also obey the EEP'.It has been shown using conservation of energy that preferred frame and preferred location affects can cause violations of the WEP[14].This goes some way toward demonstrating Schiff's conjecture,but there is as yet still no incontrovertible proof for its veracity[1].

The experiments we have just described provide very tight constraints on the WEP, the EEP, and local position invariance. It is possible to test various other aspects of relativistic gravitational theories that one may consider as 'foundational', for example the constancy of a constant of nature[20]. In our case we are interested mostly in the EEP as theories that obey the EEP are often described as being 'metric' theories of gravity, as any theory of gravity based on a differentiable manifold and a metric tensor that couples to matter, can be shown to have test particles that follow geodesics of the resulting metric space. The basics of the Riemannian geometry then tells us that at every point in the manifold there exists a tangent plane, which in cases with Lorentzian signature is taken to be Minkowski space. This allow us to recover Special Relativity at every point, up to the effects of second order derivatives in the metric, i.e tidal forces, so that the EEP is satisfied[1].Validity of the EEP can be thought of as implying that the underlying gravitational theory should be metric one[1],[21].

The Principle of Covariance

Lets recall the principle of Special Relativity, namely, all inertial observers are equivalent. The theory of General Relativity attempts to include non-inertial observers in order to cope with gravitation. Einstein argued that all observers, whether they are inertial or not, should be capable of discovering the laws of Physics[2]. If this was not true, then we would have little chance of discovering them since we are bounded in this planet, whose motion is almost certainly non-inertial. Thus Einstein proposed that all observers are equivalent. Observers are tied up with their frame of reference systems or coordinate systems, so if any observer can discover the laws of Physics, then any coordinate systems should do[2].

The situation however is different in General Relativity. In Special Relativity the metric is flat and the connection integrable, hence there exists a canonical or preferred coordinate system, namely, the Minskowksi coordinate system[2]. In a curved space-time we have a Manifold with a non-flat metric and there is no canonical coordinate system. This is just another statement of the non-existence of a global non-inertial observer. It is not so much that any coordinate system will do, but rather than the theory is invariant under a coordinate transformation. Hence we can formulate the Principle of Covariance[2],[3],[4],[5],[6] that says:

The shape of all physical equations should be the same in any arbitrary frame of reference, or equivalently, the equations of physics should have tensorial form. This principle refers to the most general case of non-inertial frames, in contrast with SR which works only with inertial frames of reference. If the Covariance Principle wasn't true then the physical equations would be different in gravitational fields and inertial-frames of reference, and hence would admit different solutions. This way the equations would predict the difference between a gravitational field and a non-inertial frame of reference and so contradict the experimental data as there is no way to distinguish between a gravitational field and a non-inertial frame of reference[2],[6].

The Principal of Minimal Coupling

The principles we have discussed so far do not tell us how to obtain field equations of systems in General Relativity when the corresponding equations are known in Special Relativity[2]. The principle of minimal gravitational coupling is a simplicity principle that essentially says we should not add unnecessary terms in making the transition from the special to the general theory. For example, in Special Relativity the energy-momentum conservation law is given[2]by:

$$\partial_b T^{ab} = 0 \tag{8}$$

The simplest generalization of the above law in General Relativity is

$$\nabla_b T^{ab} = 0 \tag{9}$$

In other words we can say that if one wants to take into account all effects of Gravity on any local physical process, described by the corresponding equations,written in the framework of Special Relativity,one should replace all partial derivatives by covariant derivatives in these equations according to the following very simple but actually very strong 'transformation'[6] $\partial \to \nabla$ or $, \to ;$.

We can now formulate the Principle of Minimal Gravitational Coupling: No terms explicitly containing the curvature tensor should be added in making the transition from Special Relativity to General Relativity[2].

The Correspondence Principle

As we stated from the outset, we are engaged with modelling, and together with any model should go its range of validity. Then any new theory must be consistent with any acceptable earlier theories within their range of validity. General Relativity must agree on the one hand with Special Relativity in the absence of gravitation and on the other hand with Newtonian gravitational theory in the limit of weak gravitational fields and low velocities in comparison with the speed of light[2].

This gives rise to the Correspondence Principle which states that: When considering the behavior of systems described by the theory of quantum mechanics or general relativity, then this behavior switches to classical mechanics for large macroscopic systems and for speeds much less than the speed of light.

The Correspondence Principle was first used by Niels Bohr back in 1913 in developing his model of the atom. However it was formulated in by Bohr in 1920 so it can be used of the modern theory of quantum mechanics.

2.2 Einstein's Theory of GR

Having considered the requirements that must be satisfied by a viable relativistic theory of gravity, let us now consider Einstein's Theory of General Relativity in particular. General Relativity satisfies all of the requirements described in the previous section, either by construction, with respect to the foundational requirements, or my trial in the case of tests of metric theories of gravity[1].

General Relativity is a gravitational theory that treats space-time as a four-dimensional manifold. The connection associated with covariant differentiation, $\Gamma^{\mu}_{\alpha\beta}$, should be viewed as an additional structure on this manifold, which in general, can be decomposed into two parts: The symmetric part and the antisymmetric part, such that[1],[2],[3],[4],[5],[6]:

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{(\alpha\beta)} + \Gamma^{\mu}_{[\alpha\beta]} \tag{10}$$

In General Relativity we take the antisymmetric part of the connection $\Gamma^{\mu}_{[\alpha\beta]} = 0$ or otherwise in the language of differential geometry, we assume that the torsion vanishes. Hence we are only left with the symmetric part of the connection, which describes the curvature of the manifold[1],[2],[3],[4],[5],[6]. Therefore we have:

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{(\alpha\beta)} \leftrightarrow \Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha} \tag{11}$$

To define distances on the manifold on also requires a metric tensor, $g_{\mu\nu}$. Along the curve γ this gives a measure for the distance[1],[2],[3],[4],[5],[6]:

$$s = \int_{\gamma} d\lambda \sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} \tag{12}$$

where λ is a parameter along the curve, $x^{\mu} = x^{\mu}(\lambda)$,and over-dots here mean differentiation with respect to λ. The metric should also be considered as an additional structure on the manifold, which is in general independent of the connection. The relationship between the connection and the metric is defined via the non-metricity tensor, $Q_{\mu\alpha\beta} \equiv \nabla_{\mu} g_{\alpha\beta}$. In General Relativity it is assumed that the non-metricity tensor vanishes as the covariant derivative of the metric tensor vanishes, and so $\nabla_{\mu} g_{\alpha\beta} = 0$.

We can now use the metric to define the Levi-Chivita connection [1],[2],[3],[4],[5],[6] which has components given by the Christoffel symbols:

$$\{^{\mu}_{\alpha\beta}\} \equiv \frac{1}{2}g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \tag{13}$$

The general form of the connection can then be shown to be given by:

$$\Gamma^{\mu}_{\alpha\beta} = \{^{\mu}_{\alpha\beta}\} + K^{\mu}_{\alpha\beta} + L^{\mu}_{\alpha\beta} \tag{14}$$

where $K^{\mu}_{\alpha\beta}$ is the contorsion tensor[1] which can be defined in terms of the antisymmetric components of the connection as:

$$K^{\mu}_{\alpha\beta} \equiv \Gamma^{\mu}_{[\alpha\beta]} - \Gamma^{\rho}_{[\alpha\nu]}g^{\mu\nu}g_{\beta\rho} - \Gamma^{\rho}_{[\beta\nu]}g^{\mu\nu}g_{\alpha\rho} \tag{15}$$

and where $L^{\mu}_{\alpha\beta}$ is defined[50]in terms of the non-metricity tensor as:

$$L^{\mu}_{\alpha\beta} \equiv \frac{1}{2}(Q^{\mu}_{\alpha\beta} - Q_{\alpha\beta}^{\ \ \mu} - Q_{\beta\alpha}^{\ \ \mu}) \tag{16}$$

As mentioned above, in General Relativity the anti-symmetric part of the connection vanishes, and so does the non-metricity tensor as it is equivalent to the covariant derivative of the metric tensor which is identically zero. This means that the $K^{\mu}_{\alpha\beta}$ and $L^{\mu}_{\alpha\beta}$ also vanish. Correspondingly, as a consequence of these two assumptions the components of the connection are uniquely given by the Christoffel symbols, and so the connection and all geometric quantities derived from it, are defined entirely in terms of the metric according to the following equation:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \tag{17}$$

The resulting set of structures, after assuming that the anti-symmetric part of the connection is zero and so does the covariant derivative of the metric tensor, is known as a Riemannian Manifold, or more accurately, pseudo-Riemannian [1],[2],[3],[4],[5],[6] in the case where the metric is not positive definite, as it required to recover special relativity in the tangent space to a point in space-time. Riemannian Manifolds have a number of useful properties including tangent vectors being parallel to themselves along geodesics, the geodesic completeness of space-time implying the metric completeness of space-time, and a particularly simple form for the contracted Bianchi Identities[1]:

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0 \tag{18}$$

We will discuss the meaning of the symbols involved in the above equation in the next part of this review. However this last equation is of great significance for Einstein's Equations.

2.3 The Einstein-Hilbert Action

The Principle of Least Action

As with most field theories, the Field Equations can be derived from the variation of the action.The Principle of least action states that: The actual path taken by a Conservative Dynamical System is an extremum of S. Where S is the action of the system and it is a functional, i.e a function of the path that is itself a function[22]. The action in classical dynamics is defined as:

$$S[x^A(t)] = \int_t^T L(x^A(t), \dot{x}^A(t)) dt \tag{19}$$

with t to be the initial time and T the final time. Where L is the Lagrangian of the system defined as the function of the the positions x^A and velocities $\dot{x}^A$ of all particles given by[22]:

$$L(x^A, \dot{x}^A) = T(\dot{x}^A) - V(x^A) \tag{20}$$

where $T = \frac{1}{2} \sum_A m_A (\dot{x}^A)^2$ is the kinetic energy and $V(x^A)$ is the potential energy. Here we provide a Proof of the Principle of Least Action from D.Tong's web-book in Classical Dynamics[22]. Consider varying a given path slightly, such that:

$$x^A(t) \to x^A(t) + \delta x^A(t) \tag{21}$$

where we fix the end points of the path by demanding that $\delta x^A(t) = \delta x^A(T) = 0$. Then the change in the action is:

$$\delta S = \delta \int_t^T L dt = \int_t^T \delta L dt = \int_t^T (\frac{\partial L}{\partial x^A}\delta x^A + \frac{\partial L}{\partial \dot{x}^A}\delta \dot{x}^A) dt \tag{22}$$

At this point we integrate the second term by parts to get:

$$\delta S = \int_t^T (\frac{\partial L}{\partial x^A} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^A}))\delta x^A dt + [\frac{\partial L}{\partial \dot{x}^A}\delta \dot{x}^A]_t^T \tag{23}$$

But the last term in the above equation vanishes as we have fixed the end points such that $\delta x^A(t) = \delta x^A(T) = 0$. The requirement that the action is an extremum implies that $\delta S = 0$ for all changes in the general path $\delta x^A(t)$. This can only happen if and only if:

$$\frac{\partial L}{\partial x^A} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^A}) = 0 \tag{24}$$

The above equation is the very well-known Euler-Lagrange Equations and hence $\delta S = 0$ if and only if the Euler-Lagrange equations hold.

The Action in GR

We have seen that the variation of the action vanishes if and only if the Euler-Lagrange Equations hold. We can now present the Einstein-Hilbert action which if we vary, and considering the vanishing of the variation of the action as above, we obtain the Einstein's Field Equations. The Einstein-Hilbert action is given[1],[2],[3],[4],[5],[6],[33]by:

$$S = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\Lambda)d^4x + \int \mathcal{L}_m(g_{\mu\nu,\psi})d^4x \tag{25}$$

where $\mathcal{L}_m$ is the Lagrangian density of the matter fields ψ and $\mathcal{L}_g = \sqrt{-g}\frac{R-2\Lambda}{16\pi G}$ is the gravitational Lagrangian density, R is the Ricci curvature scalar which can be obtained by contracting the metric tensor g_{ab} with the

Ricci curvature tensor R_{ab}, such that $R = g^{ab}R_{ab}$, and finally Λ is the Cosmological Constant that represents the energy-density of the 'empty' space and sometimes called the vacuum energy.

Let us now assume that the Ricci scalar is a function of the metric only, so that R=R(g). The variation of the action with respect to the metric tensor, as we briefly discussed above, gives Einstein's Field Equations as will see in the next part. The factors of $\sqrt{-g}$ are included in equation that describes the action to ensure that the Lagrangian densities $\mathcal{L}$ transform as scalar densities under coordinate transformations[1] i.e as:

$$\hat{\mathcal{L}} = det(\frac{\partial x^\mu}{\partial x^\nu})\mathcal{L} \tag{26}$$

under coordinates transformations $\hat{x}^\mu = \hat{x}^\mu(x^\nu)$. This property ensures that the action S is invariant under general coordinate transformations, and that the resulting tensor field equations are divergence free or otherwise the contracted Bianchi identities and energy-momentum conservation equations are automatically satisfied[1].

We have outlined here how Einstein's Field Equations can be obtained from the variation of an invariant action with respect to the metric, once it has been assumed that the space-time manifold is Riemannian. The vanishing of torsion and the non-metricity then tell us that the metric is the only independent structure at the manifold, and the invariant action principle ensures that we end up with a set of tensor field equations in which energy-momentum is conserved. Because of this formulation the WEP and EEP are satisfied identically. We will now see how Einstein's Field Equations look like after we have varied the action with respect to the metric tensor. The Field Equations are represented in tensorial form and more explicitly is a system of ten (10) non-linear differential equations [1],[7].

2.4 Einstein's Field Equations

We shall now review Einstein's Field Equations of General Relativity. It is generally accepted that GR is the most successful theory of Gravitation that when expressed mathematically produces a set of ten equations, called the Einstein's Field Equations, that describe the properties of a gravitational field surrounding a given mass. To see this we recall the geometric principle of GR, which states that gravity is nothing more than the curvature of space-time.

All laws of nature can be expressed as a certain set of differential equations and in the same way Einstein's Field Equations are a set of ten non-linear partial differential equations. This set of equations has exact solutions for some physical problems, for example Schwarzchild and Kerr solutions that describe the final collapsed state of massive bodies, and non-exact solutions for some other physical problems, such as the gravitational fields of stationary rotating stars, and the two body problem.

We make a variation of the action with respect to the metric tensor. According to GR the Einstein's Field Equations [1],[2],[3],[4],[5],[6],[7],[8] are given by the following tensor equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{27}$$

where $G_{\mu\nu}$ the symmetric Einstein's Tensor, $g_{\mu\nu}$ the symmetric metric tensor, G the Newton's constant of gravitation, c the speed of light, Λ the Cosmological Constant, and $T_{\mu\nu}$ the stress-energy tensor. However, the Einstein's tensor is also given by the following equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{28}$$

with $R_{\mu\nu}$ the Ricci Curvature tensor which can be obtained by contraction of the Riemann Curvature tensor R^a_{bad}, and R is the curvature scalar. The Riemann tensor describes an actual tidal gravitational, which is not local, and hence cannot be eliminated even in the locally inertial frame of reference.Therefore the Einstein's Field Equations are now given [1],[2],[3],[4],[5],[6],[7],[8] by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{29}$$

Finally using natural units where $G = c = 1$ and claiming that the Cosmological Constant can be absorbed in the stress-energy tensor as dark energy, then the Einstein's Field Equations read as:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{30}$$

These equations are formulated such that the energy-momentum is a conserved quantity, due to the contracted Bianchi Identity and metric-compatibility of the connection, so that special relativity can be recovered in the neighbourhood of every point in space-time, and so that the usual Newtonian Poisson Equation for weak gravitational fields is recovered in non-inertial frames kept at a fixed space-like distances from massive objects up to small corrections[1].

As we mentioned above the field equations are a set of ten generally covariant, quasi-linear second-order partial differential equations in four variables, for the ten independent components of the metric tensor. They constitute four constraint equations and six evolution equations, with the contracted Bianchi Identities ensuring that the constraint equations are always satisfied. Furthermore the conserved nature of the energy-momentum tensor $T^{\mu\nu}$ and the Riemannian nature of the Manifold ensure that the WEP and the EEP are always satisfied, that is, Massless test particles follow geodesics, and in a freely falling frame one can always choose 'normal coordinates' so that local space-time is well described as Minskowksi space-time[1].

The Vacuum Field Equations

The main condition in order to obtain the Vacuum Field Equations is for the energy-momentum tensor to vanish in the region of consideration. If the energy-momentum tensor is identically zero then the symmetric Einstein tensor is also zero[2] and hence:

$$T_{\mu\nu} = 0 \leftrightarrow G_{\mu\nu} = 0 \leftrightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{31}$$

Contracting with $g^{\mu\nu}$ and we have that:

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = 0 \leftrightarrow R - 2R = 0 \tag{32}$$

Hence we obtain the Vacuum Field Equations ($R = 0$) in the form:

$$R_{\mu\nu} = 0 \tag{33}$$

2.5 Astrophysical Tests

Deflection of Light by the Sun

First of all let us consider tests involving null geodesics. The most famous of these is the spatial deflection of the star light by the sun. In General Relativity the deflection angle, θ, of a photon's trajectory due to a mass M, with impact parameter d, is given by [1]:

$$\theta = \frac{2M}{d}(1 + cos\phi) \approx 1.75'' \tag{34}$$

where ϕ is is the angle made by the observer in the direction of the incoming photon and the direction of the mass. The first observation of light deflection was performed by noting the change in position of stars as they passed near the Sun on the celestial sphere. The observations were performed by Sir Arthur Eddington who traveled to the island of Principle near Africa to watch the Solar eclipse of May 29, 1919. According to GR, starts near the Sun would appear to have been slightly shifted because their light has been curved by its gravitational field. This effect is noticeable during an eclipse, since the Sun's brightness obscures the stars[1],[6].

The $1.75''$ is for a null trajectory that grazes the limb of the Sun. This result is famously twice the size of the effect that one might naively estimate using the equivalence principle alone[1],[23]. The tightest observational constraint to date on θ is due to Shapiro, David, Lebach and Gregory who used around 2500 days worth of observations taken over a period of 20 years. The data in this study was taken using 87 VLBI sites and 541 radio sources, yielding more than 1.7×10^6 measurements that use standard correlation and delay rate estimation procedures. The result of this[24] is around 3 orders of magnitude better than the observations of Eddington in 1919, and is given by:

$$\theta = (0.99992 \pm 0.00023) \times 1.75'' \tag{35}$$

The Perihelion Precession of Mercury's Orbit

Let us now consider tests involving time-like trajectories. The classical test of General Relativity that falls into this category is the anomalous perihelion precession of Mercury. This called a test despite the fact that it was discovered long before GR[25].In Newtonian Physics the perihelion of a test particle orbiting an isolated point-like mass stays in a fixed position, relative to the fixed stars .Adding other massive objects into the system perturbs this orbit, allowing the central mass to have to have a non-zero quadrupole moment, so that the perihelion of the test particle's orbit slowly starts to precess[1],[6].

There are a number of Solar System effects that cause the Perihelion of a planet to precess. As discussed above, the presence of other planets perturbs orbits, and so this is one of the main causes of this perturbations. In the Solar System the precession of the equinoxes of the coordinate system contribute about $5025''$ per century to Mercury's perihelion precession, while the other planets contribute about $531''$ per century. The sun also has a non-quadrupole moment, which contributes a further $0.025''$ per century. Taking all these effects into account, it still appears that the orbit of Mercury in the Solar System has an anomalous perihelion precession that cannot be explained by the available visible matter, and Newtonian gravity alone[1],[6].

The anomalous precession of the perihelion of Mercury has been calculated by many groups and a number of results are available in the paper written by S.Pireaux and J.Rozelot[26].In relativistic theories of gravity the additional post-Newtonian gravitational potentials mean that the perihelion of a test particle orbiting an isolated mass is no longer fixed, as these potential do not drop off as $\sim \frac{1}{r^2}$. There is therefore an additional contribution to the perihelion precession, which is sensitive to the relative magnitude and form of the gravitational potentials, and hence the underlying relativistic theory. For General Relativity, the predicted anomalous precession of a two body-system is given by[1]:

$$\Delta\omega = \frac{6\pi M}{p} \approx 42.98'' \tag{36}$$

where M is the total mass of the two bodies, and p is the semi-latus rectum of the orbit. The above equality is for the Sun-Mercury system, and is compatible with the observations conducted by many groups and addressed in the paper by S.Pireaux and J.Rozelot[26]. Each relativistic theory predicts its own value of $\Delta\omega$, and by comparing these observations we can therefore constrain them. This test is an additional one beyond those based on null geodesics alone, as it tests not only the 'unit curvature' of space, but also the non-linear terms in the space-time geometry, as well as preferred frame effects[1].

Spinning Objects in Orbit

Another Solar System test that involves time-like geodesics is the observation of spinning objects in orbit. These observations allow insight into an entirely relativistic type of gravitational behaviour: Gravitomagnetism. This is the generation of gravitational fields by the rotation of massive objects, and was discovered in the very early days of General Relativity by Lense and Thirring[27],[28]. The basic idea here is that massive objects should 'drag' space around them as they rotate, a concept is in good keeping with Mach's Principle. Now, in the case of GR, it can be shown that the precession of a spin vector S along the trajectory of a freely-falling gyroscope in orbit around an isolated rotating massive body at rest is given by[1]:

$$\frac{dS}{d\tau} = \Omega \times S \tag{37}$$

where

$$\Omega = \frac{3}{2}v \times \nabla U - \frac{1}{2}\nabla \times g \tag{38}$$

Here we have written the vector $g = g_{0i}$, and have taken v and U to be the velocity of the gyroscope and the Newtonian potential at the gyroscope respectively. The term $\frac{3}{2}v \times \nabla U$ is called the 'geodesic precession', and it is caused by the 'unit curvature' of the space. This effect exists independent of the massive bodies rotation. The other term in the same equation, $\frac{1}{2}\nabla \times g$, is the Lense-Thirring term, and causes the frame-dragging discussed above. The most accurate measurement of this effect claimed so far is at the level of $5\% - 10\%$ accuracy,and has been made using the Laser-Geodynamics Satellites(LAGEOS)[29],although there has been some dispute on this result[30].The Gravity Probe B mission is a more tailor made experiment which was put in orbit around the Earth between April 2004 and September 2005. The current accuracy of results from this mission are at the level of $\sim 15\%$[31],although this could improve further after additional analysis is performed.

Gravitational Redshift

The gravitational redshift was predicted by Einstein in 1907 and confirmed by Pound and Rebka in 1959. They measured the relative redshift of two sources situated at the top and bottom of Harvard University's Jefferson tower.The result was in excellent agreement with GR.The gravitational redshift was again verified by Pound and Snider in 1964 and 1965 by experiments concerning nuclear resonance and gamma radiation.

The term 'gravitational redshift' applies to electromagnetic radiation that has been sifted towards the red part of the spectrum as its wavelength has been increased and hence its energy has been decreased. In other words, photons climb out of a gravity well, and this is due to the fact that they have to transfer kinetic energy into potential energy. An analogous situation is the projectile that slows down when it rises as it has to convert kinetic to potential energy. However, in our case the speed of photons remain constant and as the photons climb out of the gravitational energy well, they have to reduce their energy by keeping their speed constant. The only way of achieving this is by reducing their frequency and hence increasing their wavelength. On the other hand when photons fall into a gravitational field they convert some of their potential energy into kinetic energy and hence they are bluesifted as their wavelength decrease.

Gravitational Waves and Binary Pulsars

A generic prediction of all known theories of gravity is the existence of gravitational waves, which are nothing more than propagating gravitational disturbances in the metric itself[1],[6]. A weak gravitational field is a small perturbation of the Galilean metric η_{ij}

$$g_{ij} = \eta_{ij} + h_{ij} \tag{39}$$

The gravitational wave is a transverse and traceless part h_{ij} of these perturbations and the plane wave has two independent states of linear polarization[6]. According to Einstein's Field Equations in empty space-time $R_{ij} = 0$, and hence it can be shown that the gravitational waves satisfy the wave equation:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})h_{ij} = 0 \tag{40}$$

While all known relativistic gravitational theories predict gravitational radiation, they do not all predict the same type of radiation as the quadrupolar, null radiation in GR. It is therefore the case that while the mere existence of gravitational radiation is not itself enough to effectively discriminate between different gravitational theories, the type of gravitational radiation that is observed,is. Briefly,one can test the speed of propagation of the gravitational waves, and a second more discriminatory test, is of the polarity of gravitational radiation[1].

At present, the highest accuracy null-observations of gravitational radiation are those of the Laser Interferometer Gravitational-wave Observatory (LIGO). The experiment has accuracy of detecting oscillations in space at a level of ~ 1 part in 10^{21},but yet has to make a positive detection. Further experiments are planned for the future using both LIGO and LISA, where positive detections of gravitational waves are expected[1].

Another way to search for gravitational waves is to look for their influence on the systems that emitted them. In this regard binary pulsars are of particular interest. Pulsars are rapidly rotating neutron stars that emit a beam of electromagnetic radiation, and was first observed in 1967[32].When the beams pass over the Earth, as the star rotates, we observe regular pulses of of radiation.The first binary pulsar PRS B1913+16 was first observed in 1974 by Russell Hulse and Joseph Taylor(1974) at Arecibo.A binary pulsar is a pulsar with a companion, often another pulsar, white dwarf or neutron star.In the above case the famous system consists of a pulsar and a neutron star. This binary system,for example,exhibits a relativistic periastron advance that is more than 30,000 times that of Mercury-Sun system. In this regard the binary pulsars provide an important compliment to the observations of post-Newtonian gravity that we observe in the solar system. Also they are a source of gravitational waves. The binary pulsars allow us to test GR in the case of a strong gravitational field, and as discussed above, the system experiences periastron advance hence the radiation is red-shifted and the orbital period decreases with time due to the gravitational radiation. Finally, neutron stars are composed of a type of matter that is of particular interest for the study of self-gravitational effects[1],[6].

3 Alternative Formulations

In the previous section we derived Einstein's Field Equations under the assumption of Riemannian Geometry, i.e assuming that the torsion vanishes and that the connection is metric compatible. In this case the metric is the only remaining geometric structure, and the only sensible thing to do is to vary the action with respect to the metric. However, we can be less restrictive in specifying the type of geometry we wish to consider.For the case of the Einstein-Hilbert Action, this usually still leads to the Einstein Field Equations. For alternative theories of Gravity this is often not the case as different variational procedures and different assumptions about the geometric structures of the manifold can lead to different field equations. We will now see a few alternative formulations but for a wide range of them is covered in [34].

3.1 The Palatini Formalism

The most well-known deviation from the metric variation approach is the Palatini procedure[1],[34],[35],[36]. Recall that when varying the Einstein-Hilbert Action, in order to derive the Field Equations, the usual approach as we have seen so far,is to vary the action with respect to the metric, after assuming that the connection depends only on the metric and the covariant derivative of the metric vanishes. This is sometimes called the Metric approach, in contrast with the Palatini approach, where one assumes that the metric and the connection are independent of each other. Although this method is generally attributed to Palatini[36], Ferraris et al(1982) argued that the Palatini approach as we know it, was in fact invented by Einstein in 1925[38].

Therefore here the connection that appears in the Riemann tensor is no longer metric compatible, but the matter is still taken to couple universally to the metric only. In the Palatini action, the metric and the connection, are considered to be two independent dynamical variables, producing two sets of Euler-Lagrange Equations. One set of equations is the Einstein Field Equations, and the other set of equations ensures that the connection is metric compatible, and therefore equals the Levi-Civita connection[1],[35],[38]. The Einstein-Hilbert Action is a function of the metric only, where the Palatini Action is a function of both the metric and the connection, and hence it can be varied with respect to both of them. As we discussed above, by varying the Palatini Action with respect to the metric[1],[35],[38] we arrive to the conclusion that the connection is indeed the Levi-Civita connection. On the other hand, when we vary the action with respect to the connection[1],[35],[38] we obtain the Einstein's Field Equations.For the Palatini approach the following are assume to hold when considering this formalism:

$$\nabla_\alpha g_{\mu\nu} \neq 0 \tag{41}$$

since the connection is not metric compatible then its components are no longer given by the Christoffel Symbols

$$\Gamma^{\mu}_{\alpha\beta} \neq \left\{ {}^{\mu}_{\alpha\beta} \right\} \tag{42}$$

in addition the curvature tensor does not have all the symmetries of the Riemann tensor, in particular:

$$\partial_\mu \Gamma^{\lambda}_{\nu\lambda} \neq \partial_\nu \Gamma^{\lambda}_{\mu\lambda} \tag{43}$$

Having discussed all the relevant issues regarding the Palatini formalism we can now write the Palatini action, which as discussed above, depends both in the metric and the affine connection[1],[34],[35],[36],[37],[38]

$$S_{pal} = \frac{1}{16\pi G} \int \sqrt{-g}(g^{\mu\nu} R_{\mu\nu}(\Gamma) - 2\Lambda) d^4x + \int \mathcal{L}_m(g_{\mu\nu}, \psi) d^4x \tag{44}$$

where $R_{\mu\nu}(\Gamma)$ is intended to indicate that the Ricci tensor here is defined with respect to the connection, and not the metric, at this stage the metric and the connection are still two independent dynamical variables. The utility of the Palatini procedure when dealing with the Einstein-Hilbert action is then that the metric compatibility of the connection is derived from the action itself and so becomes a prediction of the theory, rather than being made an assumption at the beginning. For theories of gravity other than General Relativity, however, the difference between the metric variation and the Palatini procedure is more significant: The resulting field equations are in general different[1],[35],[36],[37],[38].

The Field Equations

In the matter action there can be covariant derivatives and the only way to avoid having a matter action independent of the connection $\Gamma^{\lambda}_{\mu\nu}$ is to assume that is the Levi-Civita connection of the metric that is used for the definition of the covariant derivative.Again, we should stress once more that the underlying geometry is Pseudo-Riemannian. It is also worth noticing that this make our choice for the gravitational action even more ad hoc as now the scalar R would not be related to the curvature of space-time from a geometrical perspective[41].

As discussed above, the Palatini variation is an independent variation with respect to the metric and the connection.The easiest way to proceed with the independent variation is to express the $\Gamma' s$, as a sum of the Levi-Chivita connection of the metric tensor $g_{\mu\nu}$, and the tensor field $C^{\lambda}_{\mu\nu}$[41],[42]. Variation with respect to the $\Gamma' s$ will then be equivalent to the variation of $C^{\lambda}_{\mu\nu}$. On the boundary both the metric tensor and the C-tensor are fixed and hence by varying the action with respect to the connection we obtain from the principle of least action[41],[42]:

$$0 = \tfrac{1}{16\pi G} \int d^4x \sqrt{-g}[(-2)g^{\mu\nu}\nabla_{[\mu}\delta C^{\lambda}{}_{\lambda]\nu} + (C^{\nu\sigma}{}_{\sigma}\delta^{\mu}_{\lambda} + C^{\sigma}{}_{\sigma\lambda}g^{\mu\nu} - 2C^{\nu\mu}{}_{\lambda})\delta C^{\lambda}{}_{\mu\nu}]$$

$$+\frac{1}{16\pi G}\int d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G T_{\mu\nu})\delta g^{\mu\nu} \qquad (45)$$

The first term in the above equation is a surface term. However this time the term $\delta C^{\lambda}_{\mu\nu} = 0$ on the boundary as $C^{\lambda}{}_{\mu\nu}$ is fixed there. Coming back to the above equation and considering that the independent variations with respect to the metric and with respect to $C^{\lambda}{}_{\mu\nu}$ should vanish separately, we can see that requiring the second term to vanish corresponds to:

$$C^{\lambda}{}_{\mu\nu} = 0 \qquad (46)$$

or otherwise we have that:

$$\Gamma^{\mu}_{\alpha\beta} = \left\{{}^{\mu}_{\alpha\beta}\right\} \qquad (47)$$

so we have shown that the $\Gamma' s$ have to be the Levi-Civita connection of the metric.Therefore in the end, after we have varied the action, we still recover Einstein's Field Equations in the form: $G_{\mu\nu} = 8\pi T_{\mu\nu}$. Concluding, we have assumed that the connection is not immediately regarded to be metric compatible and we have shown that the connection is indeed the Levi-Civita connection. Given this fact, we can now conclude that R is indeed the usual Ricci curvature tensor. We have, however, assumed that initially this wasn't the case and R was not really related the the curvature of space-time from a geometric perspective.We note, that the inclusion of the cosmological constant to the gravitational part of the action reveals the standard Einstein Field Equations with a non-vanishing cosmological constant. Similarly, if we don't take into account the cosmological constant and set $\Lambda = 0$ then we end up with the same Field Equations but without the cosmological constant[41].

It should be stressed that $\Gamma^{\mu}_{\alpha\beta} = \{^{\mu}_{\alpha\beta}\}$ is now a dynamical equation or otherwise a prediction of this theory, and therefore not an assumption.Hence the Palatini formalism leads to General Relativity without the metricity condition being an external assumption. However there are problems associated with our choice for the action as the physical meaning of the independent connection is obscure, since it is not present in the matter action and it is not the one defining parallel transport Alternatively we can allow ${G^{\lambda}}_{\mu\nu}$ to be present in the matter action and to define the covariant derivative. However, even if we start from the same action the resulting theory will not be General Relativity[41].

3.2 Metric-Affine Gravity

Metric affine theories of gravity provide an interesting alternative to General Relativity due to the fact that in such theories the metric and affine connection are independent quantities, as in the case of the Palatini Formalism, and furthermore the action should include covariant derivatives of the matter fields, with the covariant derivative naturally defined using the independent connection.As a result, in metric-affine theories a direct coupling between matter and connection is also present[1],[39],[40].

Besides the standard motivation for alternatives theories of gravity, from High Energy Physics and Cosmology, metric-affine gravity has one more appealing characteristic: the connection can be left to be non-symmetric and the theory then can include torsion. This implies, that the theory can be coupled in a more natural way to matter fields, such as Fermions[41]. We note that the stress-energy tensor of the Dirac Field is not symmetric by definition and this is something that poses an extra difficulty when we attempt to couple such field to General Relativity.In fact, one might expect that at some intermediate or high energies, the spin of particles may interact with the geometry and torsion can naturally arise. Unlike General Relativity, metric-affine gravity allows for this to happen[41].

There are a number of early works in which the metric and the parallel transport defining connection, are considered as being, to some degree, independent(see [41] and references therein).In many cases, including Einstein-Cartan Theory, some part of the connection is related to the metric as we have seen so far, the non-metricity tensor. Here we will consider the case where $\Gamma^{\lambda}{}_{\mu\nu}$ is left completely unconstrained and is determined by the field equations. This approach was first considered in[39].

General Set-Up for Metric-Affine Theories

We start by defining the covariant derivative of the connection $\Gamma^{\lambda}{}_{\mu\nu}$ acting on a tensor

$$\nabla_{\mu} A^{\nu}{}_{\sigma} = \partial_{\mu} A^{\nu}{}_{\sigma} + \Gamma^{\nu}{}_{\alpha\mu} A^{\alpha}{}_{\sigma} - \Gamma^{\alpha}{}_{\sigma\mu} A^{\nu}{}_{\alpha} \tag{48}$$

It is important to stress that the position of indices must be taken very carefully into account since in this case the connection is not assumed to be symmetric. The antisymmetric part of the connection is commonly known as the Cartan-Torsion tensor, which vanishes in the theory of General Relativity, is given by:

$$S^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{[\alpha\beta]} \tag{49}$$

The failure of the connection to covariantly conserve the metric is measured by the non-metricity tensor as we have already seen in the Einstein's theory of General Relativity. The non-metricity tensor $Q_{\mu\alpha\beta}$ vanishes in GR and so does the covariant derivative of the metric tensor. The non-metricity tensor and the metric tensor are related via the equation:

$$Q_{\mu\alpha\beta} = \nabla_{\mu} g_{\alpha\beta} \tag{50}$$

Using the connection we can construct the Riemann tensor, which we have seen before, and it is of great significance as it describes an actual tidal gravitational field, and is a function of the connection and the partial derivatives of the connection, given by:

$$R^{\mu}{}_{\nu\sigma\lambda} = -\partial_{\lambda}\Gamma^{\mu}{}_{\nu\sigma} + \partial_{\sigma}\Gamma^{\mu}{}_{\nu\lambda} + \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\alpha\lambda}\Gamma^{\alpha}{}_{\nu\sigma} \tag{51}$$

Because of the limited symmetries of the Riemann tensor in metric-affine theories of gravity, we can now express the Riemann tensor in an alternative form[40]:

$$R^{\sigma}{}_{\sigma\mu\nu} \equiv \hat{R}_{\mu\nu} = -\partial_{\nu}\Gamma^{\sigma}{}_{\sigma\mu} + \partial_{\mu}\Gamma^{\sigma}{}_{\sigma\nu} \tag{52}$$

This tensor is called the homothetic curvature[40]. For a symmetric connection it is equal to the antisymmetric part of the $\hat{R}_{\mu\nu}$. The homothetic curvature is fully antisymmetric and hence when contract with the metric leads to a vanishing scalar.

As we have already discussed, in metric-affine theories of gravity the metric and connection are considered to be independent as in the case of the Palatini formalism. In this case the connection is assumed to define parallel transport and the covariant derivatives of matter fields, and hence enters the matter action[1],[40],[41].Therefore we deal with the situation where matter fields are allowed to couple not only to the metric but also to the connection, and in that sense the action takes the form:

$$S = \frac{1}{16\pi G}\int \sqrt{-g}(g^{\mu\nu}R_{\mu\nu}(\Gamma) - 2\Lambda)d^4x + \int \mathcal{L}_m(g_{\mu\nu}, \Gamma^{\mu}{}_{\alpha\beta}, \psi)d^4x \tag{53}$$

Without assuming anything about torsion or non-metricity we can find that a variation of the action with respect to the connection gives[1],[40]:

$$S^{\mu}{}_{\alpha\beta} + 2\delta^{\mu}{}_{[\alpha}S^{\nu}{}_{\beta]\nu} + \delta^{\mu}{}_{[\alpha}Q_{\beta]} - \delta^{\mu}{}_{[\alpha}\hat{Q}_{\nu]\beta}^{\ \ \nu} = 8\pi G\frac{g_{\beta\nu}}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta\Gamma^{\mu}{}_{\alpha\nu}} \tag{54}$$

with $Q_{\mu} = \frac{1}{4}Q_{\mu\nu}{}^{\nu}$ and $\hat{Q}_{\mu\alpha\beta} \equiv Q_{\mu}g_{\alpha\beta}$. This equation can be shown to be self-inconsistent for reasonable forms of matter, as the left-hand-side is invariant under projective transformations of the form $\Gamma^{\mu}{}_{\alpha\beta} \to \Gamma^{\mu}{}_{\alpha\beta} + \lambda_{\mu}\delta^{\mu}{}_{\beta}$, while there is no reason to suspect this invariance is exhibited by the matter fields[1]. Self-consistency then demands that both torsion and non-metricity to vanish leading again the the usual Einstein's Field equations when the action is varied with respect to the metric[1].

In the case of General Relativity we therefore have the lifting of a further constraint on our initial assumptions about geometry.If we allow the metric and connection to be independent, and the matter fields to couple to both the metric and the connection, then we can derive the vanishing of the torsion and non-metricity from the action itself as a set of consistency conditions. For the Einstein-Hilbert action this results in recovering the same set of field equations as with the metric and Palatini approaches. However this is not the case for alternative theories of gravity[1].

3.3 The Vierbein Formalism

The Lagrangian of general relativity is usually formulated using the components of the metric tensor as the basic field variables. Although the metric formulation is appropriate for pure gravity or gravity with bosons, the presence of spinors requires the introduction of a larger set of variables. These are the vierbein fields which describe local orthonormal Lorentz frames at each space-time point and with respect to which the spinors are defined[43].

We begin with a brief review of the definition and properties of vierbein fields. The latter are a set of four orthogonal vectors $e^{\alpha}(x)$ with $\alpha = 0, 1, 2, 3$ defined on the space-time manifold. The index α labels the independent vectors, each of which also carries a coordinate index μ in the form $e^{\alpha}_{\mu}(x)$ when expressed in component form. The non-invertible relation between these sixteen components and the ten metric components $g_{\mu\nu}(x)$ is embodied in the equations[1],[43]:

$$g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} {e_{\mu}}^{\hat{\alpha}} {e_{\nu}}^{\hat{\beta}} \tag{55}$$

$$\eta^{\hat{\alpha}\hat{\beta}} = g^{\mu\nu} {e_{\mu}}^{\hat{\alpha}} {e_{\nu}}^{\hat{\beta}} \tag{56}$$

where indices with hats correspond to a basis in the tangent space defined by the set of contravariant vectors, ${e_{\hat{\mu}}}^{\mu}$, with determinant $e = det[{e_{\hat{\mu}}}^{\mu}]$. The inverse of ${e_{\hat{\mu}}}^{\mu}$ is ${e_{\mu}}^{\hat{\mu}}$, such that ${e_{\hat{\mu}}}^{\nu} {e_{\nu}}^{\hat{\rho}} = {\delta_{\hat{\mu}}}^{\hat{\rho}}$. The use of Vierbein fields as basic variables in the usual second-order form of the action principle does not yield any information apart from the usual Einstein's Field Equations, and hence we obtain immediately $G^{\mu\nu} e^{\alpha}_{\nu} = 0$, which when multiplied by the non-singular quantity e^{ρ}_{α} gives $G^{\mu}_{\rho} = 0$[1],[43].

Clearly the Vierbein fields are only determined by the metric up to an arbitrary 'label'-space rotation by the local O(3,1)Lorentz group.The redundant components of e^{α}_{μ} specify the relation between the orthonormal frame and the local coordinate system.Corresponding to the 40 components of the metric affinity are the 24 spin connection 'rotation coefficients' $B_{\mu\alpha\beta}$ satisfying[43]:

$$\omega_{\mu\alpha\beta} = e^{\sigma}_{\alpha} e_{\beta\nu} \Gamma^{\nu}_{\mu\sigma} + e_{\beta\sigma} e^{\sigma}_{\alpha,\mu} \tag{57}$$

$$\omega_{\mu\alpha\beta} = e^{\nu}_{[\alpha}(\partial_{\mu} e_{\nu\beta]} - \partial_{\nu} e_{\mu\beta]} - e_{\mu\gamma} e^{\rho}_{\beta]} \partial_{\nu} e^{\gamma}_{\rho}) \tag{58}$$

In the Vierbein formalism the Einstein-Hilbert action can be written as[1],[43],[44]:

$$S = \int d^4x e^{\mu}_{\hat{\alpha}} e^{\nu}_{\hat{\beta}} R_{\mu\nu}{}^{\hat{\alpha}\hat{\beta}} \tag{59}$$

The spin connection $\omega_{\mu}{}^{\hat{\alpha}\hat{\beta}}$ then defines a space-time and Lorentz covariant derivative, $\mathcal{D}_{\mu}$, as:

$$\mathcal{D}_{\mu} v^{\rho}_{\nu} = \nabla_{\mu} v^{\rho}_{\nu} + \omega_{\mu\hat{\lambda}}{}^{\hat{\rho}} v^{\hat{\lambda}}_{\nu} \tag{60}$$

The curvature tensor $R_{\mu\nu}{}^{\hat{\alpha}\hat{\beta}}$ is defined in terms of the spin connection as[50]:

$$R_{\mu\nu}{}^{\hat{\alpha}\hat{\beta}} = \partial_{\mu}\omega_{\nu}{}^{\hat{\alpha}\hat{\beta}} - \partial_{\nu}\omega_{\mu}{}^{\hat{\alpha}\hat{\beta}} + \omega_{\mu}{}^{\hat{\alpha}\hat{\rho}}\omega_{\nu\hat{\rho}}{}^{\hat{\beta}} + \omega_{\nu}{}^{\hat{\alpha}\hat{\rho}}\omega_{\mu\hat{\rho}}{}^{\hat{\beta}} \tag{61}$$

Now we can make the same assumption as in the Palatini formalism and consider the spin connection and the vierbein fields to be independent. In this case we obtain two fields equation[1]:

$$\mathcal{D}_{[\mu} e^{\hat{\alpha}}{}_{\nu]} = 0 \tag{62}$$

$$G^{\alpha}_{\hat{\rho}} = e^{\alpha}_{\hat{\alpha}} e^{\mu}_{\hat{\rho}} e^{\nu}_{\hat{\beta}} R_{\mu\nu}{}^{\hat{\alpha}\hat{\beta}} - \frac{1}{2}(e^{\mu}_{\hat{\alpha}} e^{\nu}_{\hat{\beta}} R_{\mu\nu}{}^{\hat{\alpha}\hat{\beta}}) e^{\alpha}_{\hat{\rho}} = 0 \tag{63}$$

The first equation can be used to obtain the spin connection in terms of the partial derivatives of the vierbein fields, and the resulting relation implies that the spin connection is torsion-less, i.e we recover the Cartan's first structure equation: $de^{\hat{\mu}} + \omega^{\hat{\mu}}{}_{\hat{\nu}} \wedge e^{\hat{\nu}} = 0$. The second equation tells us that the vacuum Einstein's Field Equations are recovered[1].

3.4 Other Formalisms

Another interesting formulation of General Relativity is given by the Plebanski formalism[1],[45]. The action in this case is:

$$S = \int \Sigma^{AB} \wedge R_{AB} - \frac{1}{2}\Psi_{ABCD}\Sigma^{AB} \wedge \Sigma^{CD} \tag{64}$$

where upper case indices denote two component spinor indices to be raised and lowered with ϵ^{AB} and its inverse, and where the wedge product $\wedge$ acts on space-time indices, which have been suppressed.Further, the curvature 2-form R_{AB} is defined by[1]:

$$R_{AB} = d\omega_{AB} + {\omega_A}^C \wedge \omega_{CB} \tag{65}$$

If we vary the action with respect to Ψ_{ABCD} and ω_{AB} then we get that the 2-form Σ^{AB} is the exterior product some set of 1-forms that we can identify with the tetrad $\theta^{AA'}$, and that the connection ω_{AB} is torsion-free with respect to Σ^{AB}. Using this together with the variation of the action with respect to Σ^{AB} we obtain the vacuum field equations, with the metric given by $g = \theta^{AA'} \otimes \theta_{AA'}$[50].

Another interesting formulation of General Relativity is the purely affine Eddington formalism[46].We have seen so far, in other formulations of GR, that we can treat the metric as the only independent structure on the manifold, or alternatively treat the metric and connection as being two independent structures. In the Eddington formalism we treat the connection as the only independent structure on the manifold. Here the simplest way of constructing a Lagrangian density with the correct weight, and without the metric, is to take the square root of the determinant of the Ricci tensor itself[1]:

$$S = \int \sqrt{-det[R_{\mu\nu}(\Gamma)]}d^4x \tag{66}$$

To obtain the field equations we vary the action with respect to the connection which gives[1]:

$$\nabla_\gamma(\sqrt{-detR_{\alpha\beta}R^{\mu\nu}}) = 0 \tag{67}$$

The above field equations can be shown to be equivalent to the Einstein's field equations in vacuum with a cosmological constant, taking the connection to be the Levi-Civita connection. However, due to the lack of a metric in the action of this theory it is not trivial to introduce matter fields in this theory[1].

For further reading and alternative formulations of gravity the reader should look at [34] where several formulations are analysed, including the ADM Hamiltonian, the Ashtekar Hamiltonian, the CDJ Lagrangian and others.

4 Chern Simons Modified Gravity

The Chern-Simons Modified Gravity is an effective extension and a four dimensional deformation of General Relativity that captures leading order, gravitational parity violation due to the parity violating correction term given by the Pontryagin density $^*RR$ [33],[47],[48],[49],[50],[51],[52],[53],[54]. In this Section we will start by formulating the theory and by providing a pedagogical derivation of the Chern-Simons Modified Field Equations, embedding the three-dimensional CS Theory into the four-dimensional Theory of general Relativity [33],[47],[51],[52],[53],[54] and looking on various aspects of the theory. We will then discuss the application of CS Modified Gravity to CMB Polarization, and more specifically the parity violation in the Polarization of CMB, and Cosmological/Gravitational Birefrigence as consequences of parity-violating interactions. Finally we will review briefly the derivations from the Standard Model[33],[55],[56],[57],[58] and from String Theory[33],[59],[60][61], where the CS terms arise as anomaly-cancellation mechanisms, and we will also review the various astrophysical tests that have been performed so far to test the Chern-Simons Modified Gravity.

To have a general idea about the CS Theory of Gravity and its consequences in Cosmology, it is worth mentioning that as we will see, the CS correction induces parity violation, which in turn, creates two parity-violating mechanisms. The first is called Cosmological Birefrigence [33],[62],[63],[64],[65],[66] and it naturally arises with the addition of the CS term to the action, and the other is called Gravitational Birefrigence and is the prime candidate for the process of leptogenesis during inflation [33],[59],[67],[68].The Standard Model is assumed to respect parity symmetry and is always symmetric under a Charge-Parity-Time or CPT transformation. However one of the major issues in Physics is the origin of parity violations in weak interactions. While we know that all other gauge interactions respect parity, it might be the case that there is a definite handedness in cosmological scales [33]. The polarization pattern in the CMB fluctuations can leave an imprint of parity violation in the early Universe, and if parity violation can coexist on large scales, with a homogeneous and isotropic Universe then the question is how do we observe it [33],[67],[69],[70].

4.1 Formulating the Theory

Chern-Simons Modified Gravity is a four-dimensional deformation of General Relativity postulated by Jackiw and Pi [47]. The Modified Theory can be defined in terms of the action[33],[47]:

$$S = S_{EH} + S_{\theta} + S_{matter} + S_{CS} \tag{68}$$

The first term of the action is the Einstein-Hilbert term and is given by:

$$S_{EH} = k \int_V d^4x\sqrt{-g}R \tag{69}$$

The second term or the scalar field term is given by:

$$S_{\theta} = -\frac{\beta}{2} \int_V d^4x\sqrt{-g}[g^{ab}(\nabla_a\theta)(\nabla_b\theta) + 2V(\theta)] \tag{70}$$

The third term is an additional unspecified matter contribution given by:

$$S_{matter} = \int_V d^4x\sqrt{-g}\mathcal{L}_{matter} \tag{71}$$

The last term is the Chern-Simons correction term given by:

$$S_{CS} = \frac{\alpha}{4} \int_V d^4x\sqrt{-g}(\theta)^*RR \tag{72}$$

where $\mathcal{L}_{matter}$ is some matter lagrangian density that does not depend on θ, α and β are dimensional coupling constants, $k^{-1} = 16\pi G$, g is the determinant of the metric, ∇_a is the covariant derivative associated with g_{ab},

R is the Ricci Scalar given by $R = g_{ab}R^{ab}$, with R^{ab} the Ricci tensor,θ is not a constant but is a function of space-time acting as deformation function called the CS coupling field, and $\mathcal{V}$ denotes the manifold where the volume integrals are carried out. The important term in the Cern-Simons correction called the Pontryagin density ${}^*RR$[33],[47],[48],[49],[50],[51],[52],[53],[54] defined as:

$$ {}^*RR = {}^*R_b^{acd} R^b_{acd} \tag{73} $$

here ${}^*R_b^{acd}$ is th dual Riemann Tensor [33],[47],[48],[49],[50],[51],[52],[53],[54] defined as:

$$ {}^*R_b^{acd} = \frac{1}{2}\epsilon^{cdef} R^a_{bef} \tag{74} $$

with ϵ^{cdef} to be the four-dimensional Levi-Civita tensor. The Pontryagin density ${}^*RR$ is proportional to the wedge product $R \wedge R$ or the cross product in higher dimensions, but here the curvature tensor is assumed to be the Riemann Tensor. The problem that arises here is how we will determine the coupling constants, which is beyond the scope of this review. However if we leave the coupling constants unspecified so we can present generic expressions for the Modified Field Equations. If θ is a constant then the CS Modified Gravity reduces identically to GR and this is because the Pontryagin density can be expressed as the divergence [33],[48],[49],[50],[51],[52],[53],[54]

$$ \nabla_a K^a = \frac{1}{2}{}^*RR \tag{75} $$

of the Chern-Simons topological current [33],[48],[51],[52],[53],[54]

$$ K^a = \epsilon^{abcd}\Gamma^n_{\ bm}(\partial_c \Gamma^m_{\ dn} + \frac{2}{3}\Gamma^m_{\ cl}\Gamma^l_{\ dn}) \tag{76} $$

In Eq.(76) the symbol Γ refers to the Christoffel connection. Using Eq.(72) for the action of the Chern-Simons term and replacing the Pontryagin density using Eq.(75) we can now integrate the Chern-Simons part of the action by parts [33],[48] to obtain the following relationship:

$$ S_{CS} = \alpha(\theta K^a) - \frac{\alpha}{2}\int_V d^4x\sqrt{-g}(\nabla_a\theta)K^a \tag{77} $$

The first term of the RHS of Eq.(77) vanishes as it is evaluated on the boundary of the manifold. The second term vanishes as well, due to the fact that the covariant derivative of θ is zero when $\theta = constant$. Hence the Chern-Simons term equals to zero and we switch back to General Relativity.

4.2 Modified Field Equations

Derivation of the Modified Field Equations

Using the principle of least action[33],[47],[49] which states that in nature physical processes follow the most efficient course from one point to another, and starting with the following equations:

$$\delta R^b_{acd} = \nabla_c \delta \Gamma^b_{ad} - \nabla_d \delta \Gamma^b_{ac} \tag{78}$$

and also

$$\delta \Gamma^b_{ac} = \frac{1}{2} g^{bd} (\nabla_a \delta g_{dc} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac}) \tag{79}$$

we now find that:

$$\delta S = k \int_V \, d^4x \sqrt{-g} (R_{ab} - \tfrac{1}{2} g_{ab} R + \tfrac{\alpha}{k} C_{ab} - \tfrac{1}{2k} T_{ab}) \delta g^{ab}$$

$$+ \int_V \, d^4x \sqrt{-g} (\frac{\alpha}{4} {}^*RR + \beta g^{ab} \nabla_a \nabla_b \theta - \beta \frac{dV}{d\theta}) \delta\theta + \Sigma_{EH} + \Sigma_{CS} + \Sigma_\theta \tag{80}$$

in Eq.(80) the last three contributions come from the surface terms that arise due to integration by parts,C_{ab} is the C-tensor, and the term T_{ab} is the total stress-energy tensor[33],[47] and is given by:

$$T^{ab} = -\frac{2}{\sqrt{-g}} (\frac{\delta \mathcal{L}_{mat}}{\delta g_{ab}} + \frac{\delta \mathcal{L}_\theta}{\delta g_{ab}}) \tag{81}$$

where $\mathcal{L}_\theta$ is the Lagrangian density of the scalar field action, or otherwise the integrand of Eq.(70) divided by $\sqrt{-g}$. Hence the total stress-energy tensor can be split into external matter contributions T^{ab}_{mat} and a scalar field contribution [33],[47] which is given by:

$$T^{ab} = \beta[(\nabla_a\theta)(\nabla_b\theta) - \frac{1}{2}g_{ab}(\nabla_a\theta)(\nabla^a\theta) - g_{ab}V(\theta)] \tag{82}$$

Going back to Eq.(80) and having seen the $C\!-\!tensor$[33],[47],[48],[49],[50],[51],[52] which is symmetric and given by:

$$C^{ab} = v_c\epsilon^{cdea}\nabla_e R_d^b + v_{cd}{}^*R^{dabc} \tag{83}$$

where $v_a = \nabla_a\theta$ and $v_{ab} = \nabla_a\nabla_b\theta$, are the velocity and acceleration of θ. The vanishing of Eq.(80), or otherwise by the principle of least action $\delta S = 0$, leads to the modified Chern-Simons Field Equations [33],[48],[49],[50],[52] given by the following relation:

$$G_{ab} + C_{ab} = 8\pi T_{ab} \tag{84}$$

We can now see that the form of the Einstein's Field Equations is still the same apart from the new term that appears in this case, which is nothing more than the addition of the C-tensor that we discussed above. Using Eq.(28) we can get an alternative but equivalent expression for the CS Modified Field Equations:

$$R_{ab} + \frac{\alpha}{k}C_{ab} = \frac{1}{2k}(T_{ab} - \frac{1}{2}g_{ab}T) \tag{85}$$

which can be derived by noting that the C-tensor is symmetric and traceless, with T to be the trace of the stress-energy tensor given by $T = g^{ab}T_{ab}$. In the absence of matter the stress-energy tensor vanishes in the region under consideration and the Einstein's Field Equations are referred to us as Vacuum Field Equations. In addition, in the absence of matter the Ricci Tensor R_{ab} and hence the Einstein tensor G_{ab} and the C-tensor vanish and therefore the modified field equations [33],[47]and can be written as:

$$R = -\frac{1}{2k}T = 0 \tag{86}$$

The vanishing of the variation of the action leads to an extra equation of motion [33],[48] for the CS coupling field:

$$\beta g^{ab}\nabla_{\alpha}\nabla_{\beta}\theta = \beta\frac{dV}{d\theta} - \frac{\alpha}{4}{}^{*}RR \tag{87}$$

The above equation is the Klein-Gordon equation in the presence of a potential and a source term. The evolution of the CS coupling term is not only governed by its stress-energy tensor, but also from the curvature of space-time. The above equation can also be derived from the Modified CS energy-momentum equation[33] given by:

$$\nabla^{a}(G_{ab} + C_{ab}) = \frac{1}{2}\nabla^{a}T_{ab} \tag{88}$$

with the first term of the LHS of the equation to vanish by the Bianchi identities and the second term which is proportional to the Pontryagin density via [33],[48],[51],[52]

$$\nabla_{a}C^{ab} = -\frac{1}{8}v^{b*}RR \tag{89}$$

Then Eq.(88) is established by Eq.(87) provided that [33],[47],[72]:

$$\nabla_{a}T^{ab}_{mat} = 0 \tag{90}$$

This above equation is nothing more than the Strong Equivalence Principle. Alternatively, if we recall the Strong Equivalence Principle which applies to all laws of nature and is unique to Einstein's General Theory of Relativity, or more explicitly that the free-fall of an object is completely independent of its gravitational self-energy, then Eq.(88) tells us, provided that the scalar field satisfies Eq.(87) , that the Strong Equivalence Principle is satisfied since matter follows geodesics determined by the conservation of the stress-energy tensor.

4.3 Parity Violation in CS Modified Gravity

Parity Violation

Symmetries have long played a crucial role in physics. The conservation laws The conservations laws had more fundamental roots within the symmetry of the universe. Such laws as conservation of angular momentum arise from an even more fundamental requirement: Physical laws are invariant under translation and rotation. The law of conservation of parity arose from the symmetry between the left and right hands. The question of great importance is whether nature prefers left or right and vice versa. To describe more precisely the symmetry between left and right, physicists used the word parity that originated within the framework of quantum mechanics [33],[73].

In Physics a parity transformation is the flip in the sign of one spatial coordinate, and in the case of three-dimensions is the simultaneous flip in the sign of all three spatial coordinates such that:

$$P : (x, y, , z) \rightarrow (-x, -y, -z) \tag{91}$$

Then parity violation can be defined as the purely spatial reflection of the triad that defines the coordinate system[33],[73]. The operation $\hat{P}[A] = \lambda_p A$ is said to be even or parity preserving when $\lambda_p = +1$, while it is said to odd or parity violating when $\lambda_p = -1$.Hence by definition we have that $\hat{P}[\epsilon^{ijkm}] = -\epsilon^{ijkm}$ [33]. Parity transformations are slicing dependent, discrete operations, where we must specify some space-like hyperspace on which to operate. On the other hand the combined parity and time-reversal operations is a space-like operation that is slicing independent[33].

Parity violation occurs when the rate for a particle interaction is different for the mirror image of this interaction. The electromagnetic, strong, and gravitational interactions respect parity. So parity is a good symmetry for these interactions and is said to be conserved by them. On the other hand the Weak interaction does not respect parity. This was first observed in charged current interactions, or otherwise the exchange of W^+ and W^- interactions in 1956 by Madame Wu and collaborators studying the radioactive decay of isotope 60-Cobalt.

It is still unknown how parity violation arises from a unified scheme which includes all other forces, in particular gravity. In principle, the parity violation in General Relativity leads to leptogenesis by transmitting itself into Baryon-Lepton violation through primordial gravity waves. This occurs because of the gravitational Chern-Simons coupling to a pseudo-scalar field which is generated through the Green-Schwarz mechanism [33],[70]. In the Standard Model the CS correction introduces parity violation, which in turn is inspired by CP violation, therefore this is another case where we have CP violation apart from some certain types of the Weak interaction and specifically the decay of kaons. CP violation is especially intriguing, since it is believed to be the main component in order to explain the matter-antimatter asymmetry in the Universe. Conventional field theories such as the Standard Model, are always symmetric under a combined charge conjugation, parity, and time reversal transformation, also known as CPT.

The case of CS Modified Gravity

We would like to examine how the CS modification transforms under parity. General Relativity can readily be extended to have parity violation by inlcuding the Chern-Simons correction term. For homogeneous and isotropic space-times, such as the de-Sitter and FRW, this term vanishes [33],[70]. We stretch out the importance of the dependence of the CS term on the Pontryagin density ${}^*RR$ that violates parity according to:

$$
{}^*RR \rightarrow -{}^*RR \tag{92}
$$

As we said above for homogeneous and isotropic space-times the CS term vanishes, however this is not longer the case in the presence of a rolling pseudo-scalar field. Applying a parity transformation to the action, we find that it is invariant if and only if θ transforms like a pseudo-scalar, or otherwise $\hat{P}[\theta] = -\theta$ [33]. Applying such a transformation to the Modified Field Equations we find that the C-tensor is invariant if and only if the covariant velocity of θ transforms as a vector $\hat{P}[v_a] = +v_a$, or equivalently if θ is a pseudo-scalar [33].

It is worth mentioning that the properties of a solution of a theory does not necessarily have to obey the parity properties of the theory itself. As an example, Maxwell's Equations and the action do respect parity (even parity), however solutions exist where the symmetry is not respected [33]. Another example can be obtained from GR, where the theory is clearly parity-preserving, but there are solutions such as the Kerr metric and certain

Bianchi models that they violate parity [33]. If we look for parity preserving solutions such as spherically symmetric line elements then $^*RR = 0$ [33],[48],[49],[50],[51],[52] which forces θ to be constant and leads to parity-even elements that are not CS corrected. On the other hand if we look at parity-violating space-times such as the Kerr metric then the Pontryagin density will source a non-trivial CS scalar, which in turn modifies the Kerr metric through the field equations. This type of corrections tends to introduce more parity violation in the solution [33].

We have shown that the addition in the action of the CS term induces parity violation, which in turn, as we will show, creates two parity-violating interactions. The first is called Cosmological Birefringence [33],[62],[63],[64],[65],[66]and it naturally arises with the addition of the CS term to the action, and the other is called Gravitational or Amplitude Birefringence and is the main candidate for leptogenesis during inflation [33],[59],[67],[68]. The mechanism of leptogenesis is based on gravity waves during inflation. When inflation is driven by a pseudo-scalar field, the metric perturbations become birefringent.

Evidence for Parity Violation

As we discussed so far, the Standard Model respects parity symmetry and it is always symmetric under a CPT transformation. While we know that all gauge interactions, apart from the Weak, respect parity, it maybe the case that there is a definite handedness in cosmological scales. The evidence of parity violation can be found in the polarization of the CMB. A map of the CMB temperature and polarization could provide us with signatures of parity violation.

The polarization pattern in the CMB fluctuations can leave an imprint of parity violation in the early universe though a positive measurement of cross correlation functions that are not parity invariant. If parity violation on large scales can coexist with a homogeneous and isotropic universe, then the question is how do we observe it. So far it has been found that the direct signal would be undetectable in the most cases and parity violation sourced by a non-vanishing phase of a pseudo-scalar inflaton can provide all Sakharov conditions for leptogenesis [33],[67],[69],[70].

As we discussed parity violation in GR leads to leptogenesis by transmitting itself into a Baryon-Lepton violation through primordial gravity waves. This happens because there is gravitational Chern-Simons term coupling to a pseudo-scalar field which is generated through the Schwarz mechanism. The

Chern-Simons operator gives a contribution to the energy-momentum tensor leading to a suppression of the odd-parity modes in the power spectrum. If parity is violated during the inflationary period, the large scale, odd-parity, perturbations of the inflation field will experience a loss of power. This happens because the the gravitational back-reaction induces a velocity dependent potential for the primordial scalar fluctuations. At the same time the back-reaction will produce leptons. The power suppression will cease for large multipoles, which coincides with energy scales comparable to a massive right-handed neutrino. Parity violation in the early universe can tie together the two persistent anomalies in the CMB; loss of power and the alignment of low multipole moments along a preferred axis which has even mirror parity and called the 'Axis of Evil'.The 'Axis of Evil' corresponds to a direction in which global symmetries are broken [1],[69],[70],[74].

The major postulate of modern cosmology is the homogeneity and isotropy of our universe. However there have been a number of interesting claims of evidence for a preferred direction in the universe, making use of the first year results from the WMAP [1],[70],[74]. It has been suggested that a preferred direction in the CMB fluctuations may signal a non-trivial cosmic topology, a matter currently far from settled. However the preferred axis could also be the result of anisotropic expansion, possibly due to strings, walls, or magnetic fields, or even the result of an intrinsically inhomogeneous universe. As discussed above it has been found recently that the 'Axis of Evil' has a preferred frame in the WMAP data which is significantly aligned for multipoles $l = 1, 2, 3, 4, 5$ which defines an overall preferred axis. But on the other hand the so-called 'Axis of Evil' could be the result of galactic foreground contamination. For example the observations of the CMB can be contaminated by diffuse foreground emission from sources such as galactic dust and synchrotron radiation [1],[70],[74].

4.4 Chern-Simons Cosmology

As we have seen so far the implication of adding the CS term to the action is to have parity violation which in turn creates two parity violations mechanisms, Cosmological Birefringence and Gravitational or Amplitude Birefringence. Before we discuss the above two types of Birefringence we provide a brief overview of the meaning of Birefringence in Classical Physics.

Birefringence in Classical Physics

In classical physics the term Birefringence or double refraction refers to the decomposition of a ray of light into two rays, the ordinary ray and the extraordinary ray, when it passes through certain types of material, such as calcite crystals $CaCo_3$, or boron nitrate, depending on the polarisation of light. This effect can only occur if the structure of the material is anisotropic[75],[76].

Birefringence is characteristic of a material and can be formalized by assigning two different refractive indices to the material for different polarizations. The birefringence magnitude is defined by:

$$\Delta n = n_e - n_0 \tag{93}$$

with n_e and n_0 to be the refractive indices for parallel and perpendicular polarizations respectively relative to the axis of anisotropy.

Cosmological Birefringence

There are many galaxies that emit synchrotron electromagnetic radiation which is highly polarized. In the journey through cosmological distances, these plane-polarized waves pass through intergalactic magnetic fields and charged particles, which rotate the polarization plane of the waves via the Faraday rotation effect. However there is an additional rotation that is very different from Faraday rotation. The new rotation is wavelength-independent and depends only on the direction the wave moves through space, and more precisely on the angle between the direction of travel of the wave and a fixed direction in space. The amount of rotation is proportional also to the distance that the wave travels [65].

Cosmological birefringence is a wavelength-independent rotation by an angle Δ of the polarization of photons as they propagate over cosmological distances, and it is constrained by the CMB to be $|\Delta| \leq 1^\circ$ out to red-shifts of $z \leq 1100$ for a rotation that is uniform across the sky. However the rotation angle $\Delta(\theta, \phi)$ may vary as a function of position (θ, ϕ) across the sky. It has long been the subject of interest in the context of CMB where its polarization properties crucially depend on cosmological birefringence. The origin of this effect may come from either cosmic inhomogeneities or some nontrivial coupling of photons with other fields [62],[62b],[63],[64],[65],[66].The measurement of parity violation from the CMB was first discussed by Lue, Wang, and Kamionkowski [77]. They realised that the presence of the CS term naturally leads to a rotation of the plane of polarization as a CMB photon travels to the observer [33],[77].

Effects of Cosmological Birefringence in CMB

The polarization of CMB can be decomposed into two modes of opposite parity. These are, E modes or gradient components, and B modes or curl components. Primordial density perturbations produce a polarization pattern that is purely E mode at the surface of last scatter, while primordial gravitational waves, such as those from inflation, produce a B mode [62],[62b].There are at least three different types of gravitational waves: those produced during inflation and associated with the stretching of space-time modes; those produced at the violent stage of preheating after inflation; and those associated with the Goldstone modes if inflation ends via a global symmetry breaking scenario. However there maybe other mechanisms for producing B modes, apart from gravitational waves. The most widely considered is Cosmic Shear, the deflection of CMB photons due to the weak gravitational lensing by density perturbations along the line of sight will convert some of the E modes to B modes at the surface of last scatter [62],[62b].

Another possibility is the rotation of the linear polarization of the CMB as it travels from the surface of last scatter. Hence the cosmological birefringence which is driven by a scalar or quintessence field, could be responsible for converting, in the case of scalar density perturbations, scalar E-modes to vector B-modes and vice versa. In the case of tensor perturbations, such as those from gravitational waves, cosmological birefringence should mix the E and B modes [62],[62b].

Derivation of The Cosmic Optical Rotation

In order to calculate this effect, we assume a background space-time as the spatially flat FRW expanding background. On this background we will compute the cosmic optical rotation which is a measure of the Cosmological Birefringence. It is useful to take the background FRW metric[63] using conformal coordinates:

$$ds^2 = \alpha^2(-d\eta^2 + dx^2 + dy^2 + dz^2) \tag{94}$$

where η is the conformal time and $\alpha(\eta)$ the conformal scale factor. Since the electromagnetic theory is conformal invariance in four dimensions, the Maxwell's modified equations[63]coming from the non-trivial scalar field ϕ are:

$$\nabla \cdot E = 2\nabla\phi \cdot E - 2\beta\nabla\phi \cdot B \tag{95}$$

$$\partial_\eta(E) - \nabla \times B = 2(\dot{\phi} - \nabla\phi \times B) - \beta(\dot{\phi}B + \nabla\phi \times E) \tag{96}$$

$$\nabla \cdot B = 0 \tag{97}$$

$$\partial_\eta B + \nabla \times E = 0 \tag{98}$$

then the wave equation for B becomes:

$$\ddot{B} - \nabla^2 B = \dot{\phi}(-2\dot{B} + 2\beta\nabla \times B) \tag{99}$$

We assume general wave solutions of the form $B = B_0(\eta)e^{-ik\cdot x}$ and take the z-direction as the propagation direction of the electromagnetic waves. The equations for the polarization states, $b_\pm(\eta) = B_{0x}(\eta) \pm iB_{0y}(\eta)$, turn out to be:

$$\ddot{b}_\pm + 2\dot{\phi}\dot{b}_\pm + (k^2 \mp 2k\beta\dot{\phi})b_\pm = 0 \tag{100}$$

while the equation of motion for the scalar field is:

$$\ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} = \frac{e^{-2\phi}}{\omega a^2}(B^2 - E^2 + 2\beta B \cdot E) \tag{101}$$

The above non-linear coupled equations are difficult to solve exactly. We therefore look for an approximation solution to the leading order in the small ω limit. In this limit, the solution for the scalar field would be:

$$\phi = B \int \frac{d\eta}{a^2(\eta)} + C + O(\omega); \qquad \dot{\phi} = \frac{B}{a^2(\eta)} \tag{102}$$

where B, C are the constants of integration. We also assume the coupling constant β and the value of the scalar field to be very small based on the various observational constraints. From the above expressions we see that the energy-density of the scalar field is proportional to B. We therefore now that the value of this constant must be very small in order for it not to back-react to the background cosmological evolution. Since the change of $b_\pm$ is expected to be small, we estimate the optical activity using the WKB method [78]. In the long wavelength limit and for small a small coupling constant β we assume the solution of the above equation for $b_\pm$ [63]to be:

$$b_\pm = e^{ikS_\pm(\eta)}; \qquad S_\pm(\eta) = S^0_\pm + \frac{1}{k} S^1_\pm + \dots \tag{103}$$

Hence the solution based on the above anantz is:

$$S^0_\pm = \eta; \qquad S^1_\pm = -\frac{1}{2}(-2i \pm 2\beta) \int \dot{\phi} d\eta \tag{104}$$

Then from the above solution we can see that the equation for the optical rotation of the plane of polarization is:

$$\Delta = 2\beta \int_{\eta_i}^{\eta_f} \dot{\phi} d\eta = 2\beta |\phi(\eta_f - \phi(\eta_i)| \tag{105}$$

where η_i and η_f are the initial and final conformal time for the electromagnetic field. As expected, the leading contribution to the cosmic optical rotation comes from the Parity and Charge-Parity violating term.

Gravitational Birefringence

Gravitational or Amplitude Birefringence is analogous but distinct to electromagnetic birefringence. In other words, the CS modified gravity seems to prefer a specific direction, as it annihilates a certain polarization mode, and amplifies another polarization mode. We are particularly interested in the physics of the early universe and the production of gravitational waves during inflation. The mechanism of leptogenesis is based on gravity waves produced during inflation. When inflation is driven by a pseudo-scalar field the metric perturbations generated during inflation can become birefringent. We will show the main steps of the computation for the production of gravitational waves during inflation by considering the Lagrangian [68],[69] that describes gravity waves:

$$\mathcal{L} = \frac{1}{2}M_{pl}^2\sqrt{-|g|}R + F(\phi)RR^* \tag{106}$$

where $F(\phi)$ the inflation field, ϕ the pseudo-scalar, and M_{pl} the reduced Planck mass with a value of $M_{pl} = 2.44 \times 10^18 GeV$ In general metric perturbations about an FRW universe can be parameterized as:

$$ds^2 = -(1+2\phi)dt^2 + w_i dt dx^i + a^2(t)[((1+2\psi)\delta_{ij} + h_{ij})dx^i dx^j] \tag{107}$$

where ϕ, ψ, w_i, h_{ij} parameterize respectively the scalar, vector, and tensor fluctuations of the metric. For such gravity waves which are moving in the z-direction, the metric takes the form:

$$ds^2 = -dt^2 + a^2(t)[(1-h_+)dx^2 + (1+h_+)dy^2 + 2h_\times dx dy + dz^2] \tag{108}$$

where $a(t) = e^{Ht}$ during inflation and $h_+, h_\times$ are functions of t,z. It has been argued that CP violation is believed to be the main reason for the matter-antimatter asymmetry observed in the universe. The need for CP violation manifests itself in our model through the fact that a non-zero lepton generation can be achieved when $< RR^* >$ is non-vanishing. The term RR^* receives a contribution with a definite sign from gravitational fluctuations produced during inflation, which is driven by a pseudo-scalar field. In other words, CP-violation arises in our model from the inflaton field ϕ with a CP-odd component. To see the CP violation more explicitly, it is convenient to use a helicity basis:

$$h_L = \frac{1}{\sqrt{2}}(h_+ - ih_\times), \quad h_R = \frac{1}{\sqrt{2}}(h_+ + ih_\times) \tag{109}$$

Here h_L, h_R are complex conjugate scalar fields. Plugging Eq.(108) into the Eq.(106), up to the second order in h_L, h_R, we obtain [33],[68],[69] the following Lagrangian density: $\mathcal{L} = -(h_L \circ h_R + h_R \circ h_L) + [(\frac{\partial^2}{\partial z^2} h_R \frac{\partial^2}{\partial t \partial z} h_L - \frac{\partial^2}{\partial z^2} h_L \frac{\partial^2}{\partial t \partial z} h_R) + a^2(\frac{\partial^2}{\partial t^2} h_R \frac{\partial^2}{\partial t \partial z} h_L - \frac{\partial^2}{\partial t^2} h_L \frac{\partial^2}{\partial t \partial z} h_R) + Ha^2(\frac{\partial}{\partial t} h_R \frac{\partial^2}{\partial t \partial z} h_L - \frac{\partial}{\partial t} h_L \frac{\partial^2}{\partial t \partial z} h_R)]$

where $\circ$ is the operator given by: $\circ = \frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} - \frac{1}{a^2}\frac{\partial^2}{\partial z^2}$. As it can be seen from the Lagrangian, if h_L, h_R have the same dispersion relation then RR^* vanishes. Conversely a non-zero RR^* requires gravitational birefringence during inflation. We can now obtain the equations of motion [33],[68],[69] for h_L, h_R:

$$\circ h_L = -2i\frac{\Theta}{a}\dot{h}'_L; \quad \circ h_R = +2i\frac{\Theta}{a}\dot{h}'_R \tag{110}$$

$$\Theta = \frac{4}{M_{pl}^2 a^2}\frac{d}{dt}(\dot{F}a^2) \simeq \frac{4}{M_{pl}^2}(F''\dot{\phi}^2 + 2F'H\dot{\phi}) \tag{111}$$

the dot denotes a time derivative and the prime denotes differentiation of F with respect to ϕ. To obtain the above equations we have used the fact that the inflaton field is only a function of time t. In the second line for the expression of Θ we have assumed a slow-roll inflation and hence we have dropped the terms proportional to $\ddot{\phi}$. The simplest model of this kind (slow-roll inflation) is when we have a single inflation field and the pseudo-scalar ϕ as the inflaton, known as natural inflation, but it can also be incorporated to have multiple axions such as in N-inflation models.The imaginary part of this field [68],[69] (which we can call an 'axion') can couple to gravity through:

$$\Delta\mathcal{L} = F(\phi)RR^* \tag{112}$$

and $F(\phi)$ depends linearly on ϕ as:

$$F(\phi) = \frac{\mathcal{N}}{16\pi^2 M_{pl}}\phi \tag{113}$$

with $\mathcal{N}$ depending on the details of string compactification, or the 'curling up' of extra dimensions of the theory to a very small size. So far there have been five different types of string theory, but recently it was discovered that all these are versions of the M-Theory, an eleven-dimensional theory. Coming back to our model, we can see that if ϕ is constant then $\Theta = 0$, however if not equal to zero then this leads to an enhancement in the size of Θ. Working out the value of Θ we obtain [68],[69] the following relation:

$$\Theta = \frac{\sqrt{2\epsilon}}{2\pi^2}(\frac{H}{M_{pl}})^2\mathcal{N} \tag{114}$$

where the slow-roll parameter of the inflation is given by:

$$\epsilon = \frac{1}{2}(\frac{\dot{\phi}}{HM_{pl}})^2 \tag{115}$$

Derivation of Gravitational Birefringence

In order to derive the Gravitational Birefringence we need to solve the equations of motion explicitly [68],[69]. It is convenient to introduce conformal time, which runs in opposite direction with t:

$$\eta = \frac{1}{Ha} = \frac{1}{H}e^{-Ht} \tag{116}$$

The evolution equation for h_L then is:

$$\frac{d^2}{d\eta^2}h_L - 2\frac{1}{\eta}\frac{d}{d\eta}h_L - \frac{d^2}{dz^2}h_L = -2i\Theta\frac{d^2}{d\eta dz}h_L \tag{117}$$

If we ignore Θ and solve the above equation and let $h_L \sim e^{ikz}$ this becomes the equation of a spherical Bessel function:

$$\frac{d^2}{d\eta^2}h_L - 2\frac{1}{\eta}\frac{d}{d\eta}h_L + k^2h_L = 0 \tag{118}$$

for which a positive energy solution is:

$$h_L^+(k,\eta) = e^{+ik(\eta+z)}(1-ik\eta) \tag{119}$$

We now want to solve Eq.(117) and we set:

$$h_L = e^{ikz}(-ik\eta)e^{k\Theta\eta}g(\eta) \tag{120}$$

where $g(\eta)$ is a Coulomb wave-function, and we have:

$$\frac{d^2}{d\eta^2}g + [k^2(1-\Theta^2) - \frac{2}{\eta^2} - \frac{2k\Theta}{\eta}]g = 0 \tag{121}$$

The above equation is the Schrodinger's equation for a particle with $l = 1$ in a weak Coulomb potential. When $\Theta = 0$ the Coulomb term vanishes and we get the spherical Bessel function. For h_L the Coulomb term is repulsive whereas for h_R, with the opposite sign of the Θ term, the Coulomb potential is attractive. The plane-wave solutions can be written [77] as:

$$e^R_{\mu\nu}e^{\frac{2}{M_{pl}^2}F''\dot{\phi}^2 kt}e^{-ikt+ikz} \tag{122}$$

$$e^L_{\mu\nu}e^{-\frac{2}{M_{pl}^2}F''\dot{\phi}^2 kt}e^{-ikt+ikz} \tag{123}$$

Where $e^R_{\mu\nu}, e^L_{\mu\nu}$ are the polarization tensors for right and left-handed polarized waves, respectively. Hence this leads to an attenuation of the left-handed gravitational waves and an amplification of the right-handed gravitational waves in the early universe. This effect is what we call gravitational birefringence and it can also be seen in terms of the intensity [68] of radiation in each polarized wave:

$$I^{\pm}_{CS} = I^{\pm}_{E}\frac{1}{(1\pm\chi)^2} \tag{124}$$

where $I^{\pm}_{CS}$ are intensities in two polarized waves in the modified gravity, $I^{\pm}_{E}$ are intensities in Einstein's gravity, and χ is proportional to the suppressing parameter of CS corrections. In Einstein's gravity both polarizations carry equal intensities, in contrast in CS gravity the two polarizations carry different intensities.

Gravity Leptogenesis

As discussed in the previous section leptogenesis is based on gravity waves during inflation. One of the greatest puzzles in astrophysics is why there is an excess of matter over antimatter in the universe, and as we have seen CP violation is the main candidate for the matter-antimatter asymmetry. We have shown that the CS modified gravity has a preferred direction with respect to the gravity waves that are produced during inflation. The recent WMAP data [33],[59],[68],[69] shows that the baryon density, expressed as baryon number over photon number, is:

$$\frac{n_b}{n_\gamma} = (6.5 \pm 0.4) \times 10^{-10} \tag{125}$$

However in the standard model such high energy cannot be found, since the best estimate [33],[59],[68],[69],[79] provides:

$$\frac{n_b}{n_\gamma} < 6 \times 10^{-27} \tag{126}$$

The conditions [33],[59],[68],[69], for generating the matter-antimatter asymmetry are the following:

- Violation of baryon number
- Charge-Parity (CP) Violation
- Violations must occur during the era of Thermal Equilibrium

In the weak interactions contain processes which convert baryons to leptons and vice-versa. They are activated at energies greater than 1 TeV. This implies that baryon asymmetry can be viewed through the creation of net lepton number at high temperature through out-of-equilibrium and CP-violating processes. Such scenario is called leptogenesis. The lepton number current J^μ_{lepton} and hence the total fermion number current has a gravitational anomaly [33],[59],[68],[69] in the Standard Model through a term:

$$\partial_\mu J^\mu_{lepton} = \frac{N}{16\pi^2} {}^*RR \tag{127}$$

We can clearly see that the anomaly is proportional to the Pontryagin density and is a consequence of an imbalance between the left and right-handed leptons. Where $N = N_L - N_R$, and has a value of 3 in the Standard

Model. If the CS correction is non-zero during some era $t_1 < t < t_2$ then the lepton number [59],[68],[69] is:

$$n_L = \frac{N}{16\pi^2}\int_{t_1}^{t2} {}^*RRdt \tag{128}$$

The authors of [68],[69] calculated this integral and found and estimate for the lepton number which is in agreement with the recent WMAP data:

$$\frac{n_b}{n_\gamma} \sim 10^{-10} \tag{129}$$

According to the recent WMAP observations, the scalar metric perturbations generated during inflation have a size that give off density fluctuations with $\frac{\delta\rho}{\rho} \sim 10^{-5}$. On the other hand if we look at Eq.(129) we can clearly see that:

$$\frac{n_b}{n_\gamma} \sim (\frac{\delta\rho}{\rho})^2 \tag{130}$$

The above result could be a simple numerical accident or there might be some underlying deeper physics.

4.5 The Many Faces of Chern-Simons Gravity

We will briefly discuss the natural emergence of the CS corrections in the Standard Model of Particle Physics and in String Theory, as anomaly-cancellation mechanisms for the gravitational ABJ anomaly and the Green Schwarz anomaly respectively. We will also see that the effective 4D action for type II string theory yields exactly the same equations as CS modified gravity.

Standard Model

The first place we encounter the CS invariant is in the gravitational anomaly of the Standard Model of Particle Physics. In the Standard Model, the CS correction introduces parity violation, which in turn is inspired by CP violation, where CS terms act as an anomaly-cancelation mechanism. In particular, CS terms are necessary to cancel mixed anomalies between anomalous and non-anomalous U(1) groups [59],[80].

An anomaly describes a quantum mechanical violation of a classically conserved current. According to Noether's theorem, invariance under a classical continuous global symmetry group G implies the conservation [33] of a global current j_a^A, with A labeling the generators of the group G:

$$\partial_a j^{aA} = 0 \tag{131}$$

An anomaly [33] is a quantum correction to the divergence of the current such that:

$$\partial_a j^{aA} = \mathcal{A}^{\mathcal{A}} \tag{132}$$

In this section we will omit the derivation of the violation of the U(1) axial current, also known as ABJ anomaly, a global anomaly in the Standard Model, and instead we will present a derivation to show how anomalies are exactly canceled with the addition of CS terms in the next section. The derivation of the ABJ can be found in [55],[56]. The equation of the ABJ anomaly [12],[13],[33] is the following:

$$\partial^a j_a^A = -\frac{1}{8\pi^2}\epsilon^{abcd} F_{ab} F_{cd} \tag{133}$$

The above equation of the ABJ anomaly also applies for the gravitational anomaly. Replacing the electromagnetic field tensor with the Riemann curvature tensor we obtain the gravitational ABJ [33] anomaly:

$$D^a j_a^A = -\frac{1}{384\pi^2}\frac{1}{2}\epsilon^{abcd} R_{abef} R_{cd}^{ef} \tag{134}$$

We can see that the RHS of the above equation is proportional to the Pontryagin density $^*RR$. The gravitational ABJ anomaly can be canceled by adding the appropriate counter term in the action, which in turn implies the addition of the CS modification in the Einstein-Hilbert action.

We now present a toy model [59],[81] that consists of a chiral gauge theory with only two U(1) groups. One is anomalous with gauge field A_μ and field strength $F^A_{\mu\nu}$, the other non-anomalous with gauge field Y_μ and field strength $F^Y_{\mu\nu}$. Both are merited with charge operators. Under gauge transformations:

$$A_\mu \rightarrow A_\mu + \partial_\mu \epsilon; \quad Y_\mu \rightarrow Y_\mu + \partial_\mu \xi \tag{135}$$

Using differential form notation, the one-loop effective action [59],[82] is transformed as:

$$\delta S_{one-loop} = \int d^4x[\epsilon(\frac{1}{3}c_3 F^A \wedge F^A + c_2 F^A \wedge F^Y + c_1 F^Y \wedge F^Y) + \xi(c_2 F^A \wedge F^A + c_1 F^A \wedge F^Y)] \tag{136}$$

where c_1, c_2, c_3 are constants obtained by tracing combinations of charge operators. The classical action [83] is given by:

$$S_{axion} = \int d^4x[-\frac{1}{4g_Y^2}(F^Y)^2 - \frac{1}{4g_A^2}(F^A)^2 + (d\alpha + MA)^2 + \alpha(d_3 F^A \wedge F^A + d_2 F^A \wedge F^Y + d_1 F^Y \wedge F^Y)] \tag{137}$$

where d_1, d_2, d_3, M are constants. Under a gauge transformation α transforms as:

$$\alpha \rightarrow \alpha - M\epsilon \tag{138}$$

and it is assumed that α does not shift under non-anomalous gauge transformations parameterized by ξ. Then, the variation of the the action S is given by:

$$\delta S_{axion} = -\int \epsilon(d_3 F^A \wedge F^A + d_2 F^A \wedge F^Y + d_1 F^Y \wedge F^Y) d^4x \tag{139}$$

we now re-visiting the CS action that we have seen in this review but we present it in differential form notation:

$$S_{CS} = \int Y \wedge A \wedge (d_4 F^A - d_5 F^Y) d^4x \tag{140}$$

where d_4, d_5 are constants. The variation of the CS action added to the variation of the action gives us: $\delta S_{axion} + \delta S_{CS} = -\int \epsilon (d_3 F^A \wedge F^A + (d_2 - d_4) F^A \wedge F^Y + (d_1 + d_5) F^Y \wedge F^Y) d^4x$

$$-\int \xi (d_4 F^A \wedge F^A - d_5 F^Y \wedge F^A) d^4x \tag{141}$$

Putting $d_1 = 2c_1, d_2 = 2c_2, d_3 = \frac{1}{3}c_3, d_4 = c_2, d_5 = -c_1$ and comparing Eq.(82) and Eq.(87), we have that:

$$\delta S_{one-loop} + \delta S_{axion} + \delta S_{CS} = 0 \tag{142}$$

Hence the anomalies are exactly cancelled with the addition of the CS term and therefore the CP violation in the Standard Model naturally requires the existence of CS terms, which in turn leads to a modification of our current model of gravity.

String Theory

The CS correction term arises naturally in String Theory as we will see. In 10D supergravity the Green-Schwarz anomaly is cancelled by the CS correction. In fact, the slope of the expansion of the string theory yields the Einstein action as well as corrections of higher order in curvature. In order for this expansion to be ghost free, the quadratic term must be the Gauss-Bonet combination. The CS gravity term requires the presence of the Gauss-Bonet term for supersymmetry, and conversely, supersymmetrising the Gauss-Bonet term requires the CS term[59]. Starting with the effective string action for type II string theory

$$S = \int \sqrt{-g} (\frac{1}{16\pi} R - \alpha H_{abc} H^{abc} + \ldots) d^4x \tag{143}$$

where H_{abc} is the Kalb-Ramond 3-form field strength, α is a constant with units of length squared, and several terms are neglected, including the Gauss-Bonet terms. In differential form notation the Kalb-Ramond field is written as:

$$H = \frac{1}{3}dB + \omega_{CS} \tag{144}$$

where B is the Kalb-Ramond 2-form field ω_{CS} the 3-form field is the Chern-Simons term. The part of the action that involves the 2-form KR field B is:

$$S_B \propto \int (H\wedge^* H - \omega_{CS}\wedge^* \omega_{CS})d^4x = \int (\frac{1}{9}dB\wedge^* dB + \frac{1}{3}dB\wedge^* \omega_{CS} + \frac{1}{3}\omega_{CS}\wedge^* dB)d^4x \tag{145}$$

where * denotes the Hodge dual. Its variation in B implies the equation of motion for H:

$$d^* H = 0 \tag{146}$$

which shows that $^*H$ is closed. Hence locally there exists a pseudoscalar b, called the Kalb-Ramond axion or universal axion, such that:

$$H = {}^* db \tag{147}$$

$$H_{abc} = \epsilon^k_{abc}\nabla_k b \tag{148}$$

Varying the action we obtain:

$$G_{\mu\nu} = 8\pi[T_{\mu\nu} - \frac{1}{3}\nabla_\alpha(\epsilon_{\beta\gamma\delta(\mu}[\nabla^\beta b]R^{\alpha\gamma\delta}_{\nu)})] \tag{149}$$

where $T_{\mu\nu}$ is the stress-energy tensor corresponding to the pseudo-scalar b. Using the Bianchi Identity we can show that:

$$\nabla_\alpha(\epsilon_{\beta\gamma\delta(\mu}[\nabla^\beta b]R^{\alpha\gamma\delta}_{\nu)}) = 2C_{\mu\nu} \tag{150}$$

where $C_{\mu\nu}$ is the Cotton-York tensor. We adjust the constants by taking $b = \frac{3}{8\pi}f(\theta)$, where $f(\theta) = \frac{N}{\psi^2 m_p}\theta$, N a dimensionless number, θ the axion field that controls the CS correction, ψ the string scale, and $m_p = 2.4 \times 10^{18} GeV$ is the reduced Planck mass. Finally the previous equation becomes:

$$G_{\mu\nu} = C_{\mu\nu} + 8\pi T_{\mu\nu} \tag{151}$$

and this is exactly what we have found so far,the Modified Einstein's Field Equations under the presence of a Chern-Simons correction term. Hence in string theory the CS term arises naturally.

4.6 Astrophysical Tests

All tests of the Chern-Simons modified gravity have been performed with astrophysical observations and concern the non-dynamical framework[33], otherwise the framework in which the coupling constant β is set to zero and hence the scalar field does not evolve dynamically but it instead externally prescribed.After Alexander and Yunes[84],[85]realised that the modified theory predicts an anomalous precession effect, Smith et.al.[86] tested the non-dynamical model with canonical scalar using LAGEOS[87] and Gravity Probe B observations[88],placing the the first weak bound on the CS scalar. Then Konno et.al.[89] proposed that the CS correction could be used to explain the flat rotation curves of the galaxies, which in turn will give another constraint on the non-dynamical theory for non-canonical θ. Recently, Yunes and Spergel[90] used double binary pulsar data to place a bound on the non-dynamical model with canonical CS scalar that is eleven orders of magnitude stronger than the Solar System one.

Binary Pulsar Test

Non-dynamical CS Modified Gravity has been shown to modify only the gravitomagnetic sector of the metric[1], which does not influence most astrophysical processes. This is particularly true outside the Solar System, where stars inside galaxies will have random-oriented velocities that will lead to vanishing averaged CS correction.On Cosmological scales, the CS correction is not relevant since, for example, the equations of structure formation are not corrected[33].

However some astrophysical precesses are CS corrected, such as the formation of accretion discs around protoplanetary systems. In this case and although the CS correction is non-zero[33], the measurements are difficult as the correction would be greatly suppressed by the almost negligible compactness of such systems. On the other hand, a more interesting case deals with double binary pulsars, such as the PRS JO737-3039A/B[33],[91], which consists of two rapidly rotating stars orbiting each other.The mass of typical neutron star is approximately $1.4M\odot$ and its radius of the order of 10km, which implies a significant large compactness, and this leads to very strong gravitational fields that can be used to test GR[33],[92].

Smith et.al,[86] considered double binary pulsars and modeled their orbital evolution via a study of a compact object in the background of a rotating homogeneous sphere. The motion of this binary system is determined by the four acceleration $\vec{a} = -4\vec{v} \times \vec{B}$, with v the velocity of one of the pulsars and B the gravitomagnetic field of the other. To leading order in $\dot{\theta}$ the CS correction to the gravitomagnetic field is:

$$\vec{B}_{cs} = \frac{c_0}{r} \cos\xi(r)[\vec{j} - \tan\xi(\vec{j} \times \hat{r}) - (\vec{j} \cdot \hat{r})\hat{r}] \tag{152}$$

where $\xi(r) = \frac{2kr}{\dot{\theta}\alpha}, c_0 = \frac{15\alpha\dot{\theta}}{4kR\sin\xi(R}), \hat{r} = \frac{\vec{r}}{r}, \vec{j} = \frac{\vec{J}}{R^2}$. On the other hand the only Keplerian parameter that is CS corrected is the radial acceleration $a_r = \vec{r} \cdot \hat{r}$. The CS corrected radial acceletion is given by:

$$a_r^{cs} = -4c_0\dot{u}\dot{v}[\cos i \cos\xi r + \sin i \cos u \sin\xi(r)] \tag{153}$$

with i the inclination angle, $\dot{u} = \dot{f} + \dot{\omega}$, where ω is the argument of the perigee and f is the true anomaly.Finally the only post-Keplerian parameter that is CS corrected to leading order is $\dot{\omega}$, or the rate of change of perigee, in the non-dynamical theory with canonical CS scalar and all parameters can be obtained by [93].The CS correction to the average rate of change of the perigee is given by:

$$< \dot{\omega} >_{cs} = \frac{15J\dot{\theta}}{2a^2eR} X \sin(\frac{2kR}{\alpha\dot{\theta}}) \sin(\frac{2ak}{\alpha\dot{\theta}}) \tag{154}$$

Yunes and Spergel[90], used the calculations obtained by M.Kramer et.al[93] to place a strong bound on the non-dynamical framework. Using the relevant system parameters of PSR J0737-3039A/B, the CS scalar was constrained to be:

$$\dot{\theta} \leq 4 \times 10^{-9} km \tag{155}$$

which is 10^{11} times stronger than the current Solar System constraints. A similar analysis in the full, strong-field dynamical formalism is still lacking and is a subject for further development.

Galactic Rotation Curves

A galaxy rotation curve is the plot of the orbital velocity v of star in a galaxy against their distance to the galactic centre. According to Newtonian mechanics the orbital velocity should obey a square-root fall-off $v \propto r^{-\frac{1}{2}}$. We would expect the orbital velocity of stars to decrease as the distance from the galactic center increases according to Newton's Law, however Rubin and Ford[94],[95] found that the galaxy rotation curves flattens with distance, hence stars revolve at constant speeds over a large range of distances from the center of the galaxy. This is turn implies the existence of an additional, non-visible type of matter, or dark matter otherwise.

The only galactic study that has been carried out by Konno et.al[89], is related to the flat-rotation curves of galaxies.They have attempted to explain the flatness of rotation curves through the non-dynamical CS modified gravity with $\alpha = -lk$ and $\beta = 0$, and a non-canonical CS scalar. Their result is the following:

$$u = \pm\sqrt{\frac{M}{r} + \frac{C_2}{2} + O(J)^2} \tag{156}$$

with C_2 a constant that depends on the spin angular momentum. This result is to be contrasted with the Kerr Metric, for which $u = \sqrt{\frac{M}{r}} - \frac{J}{r^2}$.Another solution was presented by Yunes and Pretorius in[48] for the non-dynamical formulation, which is:

$$v_\phi \sim \sqrt{\frac{M}{r}} - \frac{M}{r^2} \tag{157}$$

Briefly speaking, the non-dynamical framework does not suggest that either of the two solutions presented here is more or less valid than the other. The freedom of choice of the CS scalar leads to two different observables and points at an incompleteness of the framework. This observation together with the constraints of the non-dynamical framework, creates reasonable doubts on the validity of the CS correction as an explanation of the flatness of galaxy rotation curves[33].

Gravitational Wave Tests

As we have seen the main effect of the CS correction on the propagation of gravitational waves is an amplitude or gravitational birefringence. Several tests have recently been proposed of the CS modified gravity with gravitational waves. All such tests have so far concentrated on waves generated by binary systems,where the CS correction arises due to the propagation of waves[1].

Alexander et.al,[96]have proposed a gravitational wave test of non-dynamical CS modified gravity with a generic CS scalar though the space-born gravitational wave detector LISA[97],[98],[99],[100].The sources in mind are supermassive black hole binaries at red-shifts $z < 30$. In order to determine how good of a constraint LISA could place on CS modified gravity, one would have to carry out a full covariance matrix analysis, including all harmonics in the signal amplitude,since the CS correction affects precisely this amplitude[33].

One can obtain an order of magnitude estimate by making the following assumptions: First, place two GW detectors in the process of GW detection, such that one can reconstruct left and right polarised amplitudes. Second, model the noise as white, with one-sided spectral noise density S_0. Third, we focus the attention on the covariance (Fisher) matrix Γ_{ij} related to the parameters that affect the amplitude of the GW signal neglecting the phase parameters[101],[102].

The advantages of the LISA gravitational wave detection include a possible constraint on θ five orders of magnitude better than the Solar System tests. A GW detection also constraints a different sector of the modified theory, since it samples the temporal evolution of the CS scalar, instead of its local value. This is because a GW detection really constraints the evolution of the CS scalar from the time of emission of the GW to its detection on earth.

Another interesting test of the CS modified gravity can be performed using gravitational waves emitted by extreme-mass ratio in-spirals or binary black hole mergers[103].These systems sample the strong-gravitational regime of space-time, in which the CS correction is enhanced as shown by the CS modified Kerr solution[48].The generation of gravitational waves would then also CS modified, not only due to the corrections of the background metric, but also due to the fact that the CS scalar must carry energy-momentum away from the system. Even ignoring the latter, Sopuerta and Yunes[103] have shown that the background modifications lead to extreme and intermediate ratio in-spirals, whose waveforms are sufficiently distinct from their GR counterpart to allow for a test of the radiative sector of the dynamical theory over a few-month integration period[33].

Solar System Tests

The non-dynamical modified theory has been so far only through frame-dragging Solar System experiments. The non-dynamical CS modification induces anomalous precession effects. Smith et.al[86]studied the anomalous precession using the values of $\alpha = -\frac{l}{3}$ and $\beta = 1$. Precession is a term that refers to the change in rotation 3-vector of a spinning object.There are two types[33] of precession that can be distinguished:

- Torque-Free
- Torque-Induced

The former corresponds to situations in which the spin angular momentum is not co-aligned with the axial Killing vector, or otherwise a vector field on the Riemannian manifold that preserves the metric[33],and the latter is also known as gyroscopic precession, occurring in situations where there is an additional torque, such as that of a gyroscope. Gyroscopic precession can be studied in the Newtonian framework but relativity adds three additional corrections[33]:

- Thomas Precession
- De-Sitter or Geodesic Precession
- Lense-Thirring Precession

Thomas precession is due to the observer's non-inertial rotating frame and is an additional Special Relativity correction. De-Sitter precession is a GR effect that accounts for Schwarchild-like deviations from flat space-time, and Lense-Thirring precession another GR correction due to the gravitomagnetic sector of the Kerr metric[33].

Consider first the motion of a test body in the external field of a CS spinning source. In such a field the test body will experience Lense-Thirring precession which will be different in CS gravity with respect to GR[104].Smith et.al.[86], studied the secular time variation of the longitude of the ascending node $\dot{\Omega}_{orb}$ in the non-dynamical modified theory and found that:

$$\dot{\Omega}_{obs} = \dot{\Omega}_{GR} + \dot{\Omega}_{CS} \tag{158}$$

where the GR Lense-Thirring drag is given by:

$$\dot{\Omega}_{GR} = \frac{2GJ}{\alpha^3(1-\epsilon^2)} \tag{159}$$

with ϵ to be the eccentricity, J the magnitude of the spin angular momentum of the central body. The CS correction is given by:

$$\dot{\Omega}_{CS} = \frac{15\alpha^2}{R^2} j_2(m_{cs}R) y_1(m_{cs}\alpha) \tag{160}$$

where α is the semi-major axis, R is the earth's radius, and j_2, y_1 are the spherical Bessel functions of the first and second kind. The LAGEOS experiments have measured $\dot{\Omega}$ and found that it is in agreement with General Relativity up to an experimental error, which allows to test for a non-dynamical Chern-Simons gravity. Smith et.al.[86] estimated also the constraint on the CS scalar and found that $\dot{\theta} \leq 3 \times 10^{-20}$km.

The next type of precession that is affected by the CS gravity is gyroscopic precession. Consider a gyroscope with spin angular momentum S in circular orbit around the Earth. The rate of change of S is geiven by[33]:

$$\dot{S}^i = 2\epsilon^{0ijk} B_j B_k \tag{161}$$

where B_i is the gravitomagnetic field. The precessional angular velocity is then given by $\dot{\Phi} = \frac{|\dot{S}^i|}{|S^i|}$, which is CS corrected by:

$$\frac{\dot{\Phi}_{CS}}{\dot{\Phi}_{GR}} = 15\frac{\alpha^2}{R^2} j_2(m_{cs}R)[y_1(m_{cs}\alpha) + m_{cs}\alpha y_0(m_{cs}\alpha)] \tag{162}$$

where $\dot{\Phi}_{GR}$ is the GR prediction and R is the distance from the centre of the Earth to the gyroscope. Given an experimental verification of the Lense-Thirring effect, we could place a constraint on the CS scalar with the LAGEOS satellites.Gravity Probe B was designed to measure this effect to percentage accuracy[35]but since it was launched it faced certain difficulties that might degrade its accuracy.Smith et.al.[33] studied the studied the possibility that Gravity Probe B could place a stronger constraint than the LAGEOS satellites but this wasn't the case.The estimates of their group revealed that the CS modification in the gyroscopic precession leads to non-boundary corrections to the gravimagnetic field, a subject for further study.

5 f(R) Theories of Gravity

5.1 Introduction

General Relativity is widely accepted as a fundamental theory to describe the geometric properties of space-time. In a homogeneous and isotropic space-time the Einstein field equations give rise to the Friedmann equations that describe the evolution of the universe. In fact,the standard big-bang cosmology based on radiation and matter dominated epochs can be well described within the framework of General Relativity[112].

However, the rapid development of observational cosmology which started from 1990s shows that the universe has undergone two phases of cosmic acceleration. The first one is called inflation which is believed to have occurred prior to the radiation domination. This phase is required not only to solve the flatness and horizon problems plagued in big-bang cosmology, but also to explain a nearly flat spectrum of temperature anisotropies observed in Cosmic Microwave Background. The second accelerating phase has started after the matter domination. The unknown component giving rise to this late time cosmic acceleration is called dark energy.The existence of dark energy has been confirmed by a number of observations such as supernovae Ia,large-scale structure, baryon acoustic oscillations[112].

These two phases of cosmic acceleration cannot be explained by the presence of standard matter whose equation of state $w = \frac{P}{p}$ satisfies the condition $w \geq 0$. Here P and p are the pressure and the energy density of matter, respectively. In fact, we further require some component of negative pressure, with $w < -1/3$, to realize the acceleration of the universe. The cosmological constant Λ is the simplest candidate of dark energy, which corresponds to w = -1. However, if the cosmological constant originates from a vacuum energy of particle physics, its energy scale is too large to be compatible with the dark energy density. Hence we need to find some mechanism to obtain a small value of Λ consistent with observations. Since the accelerated expansion in the very early universe needs to end to connect to the radiation-dominated universe, the pure cosmological constant is not responsible for inflation. A scalar field with a slowly varying potential can be a candidate for inflation as well as for dark energy[1],[41],[111],[112].

Although many scalar-field potentials for inflation have been constructed in the framework of string theory and super-gravity, the CMB observations still do not show particular evidence to favor one of such models. This situation is also similar in the context of dark energythere is a degeneracy as for the potential of the scalar field or quintessence field due to the observational degeneracy to the dark energy equation of state around $w = -1$. Moreover it is generally difficult to construct viable quintessence potentials motivated from particle physics because the field mass responsible for cosmic acceleration today is very small[112].

While scalar-field models of inflation and dark energy correspond to a modification of the energy-momentum tensor in Einstein equations, there is another approach to explain the acceleration of the universe. This corresponds to the modified gravity in which the gravitational theory is modified compared to GR. The Lagrangian density for GR is given by $f(R) = R - 2\Lambda$, where R is the Ricci scalar and Λ is the cosmological constant (corresponding to the equation of state w =-1).The presence of Λ gives rise to an exponential expansion of the universe, but we cannot use it for inflation because the inflationary period needs to connect to the radiation era. It is possible to use the cosmological constant for dark energy since the acceleration today does not need to end. However,if the cosmological constant originates from a vacuum energy of particle physics, its energy density would be enormously larger than the todays dark energy density. While the Λ Cold Dark Matter model ($f(R) = R - 2\Lambda$) fits a number of observational data well,there is also a possibility for the time-varying equation of state of dark energy[1],[41],[111],[112].

5.2 f(R) Models

Fourth-order theories of gravity have a long history, dating back to as early as 1918,only a few years after the first papers on General Relativity by Einstein. These theories generalise the Einstein-Hilbert action by adding additional scalar curvature invariants to the action, or by making the action a more general function of the Ricci scalar then the simple linear one that leads to Einstein's equations. Here we consider the latter of these options, a choice that leads by Lovelock's theorem to fourth-order field equations for anything except the addition of a constant term to the gravitational Lagrangian. Such theories, generically referred to as f(R) theories of gravity, have been intensively studied, and have a number of reviews dedicated to them.This interest was stimulated in the 1960s, 70s and 80s by the revelations that the quantization of matter fields in an un-quantized space-time can lead to such theories, that f(R) theories of gravity can have improved renormalisation properties,and that they can lead to a period of accelerating expansion early in the Universe's history. More recently they have been of considerable interest as a possible explanation for the observed late-time accelerating expansion of the Universe as we discussed above[1],[41].

The f(R) generalisations of Einstein's equations are derived from a Lagrangian density of the form[1]:

$$L = \sqrt{-g} f(R) \tag{163}$$

where the factor of $\sqrt{-g}$ is included, as usual, in order to have the proper weight. This is clearly about as simple a generalisation of the Einstein-Hilbert density as one could possibly conceive of. The field equations derived from such an action are automatically generally covariant and Lorentz invariant for exactly the same reasons that Einstein's equations are. Unlike the Einstein-Hilbert term, however, the field equations that one obtains from the least action principle associated with Eq.(163) depend on the variational principle that one adopts[1]. Different possibilities are the metric variation where the connection is assumed to be the Levi-Civita one, the Palatini approach in which Eq.(163) is varied with respect to the metric and connection independently, and the metric-affine approach in which the same process occurs but the matter action is now taken to be a functional of the connection as well as the metric.

Hence the Einstein Hilbert Action[1],[33],[41] that we have already seen:

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \tag{164}$$

becomes [1],[33],[41],[111],[112],[113],[114],[115]:

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) \tag{165}$$

The above action provide us with the simplest modification[1]of GR, in which the Lagrangian density is just an arbitrary function of the Ricci Curvature Scalar.

Toy Models

The Standards Einstein gravity maybe modified at low curvature, by including the terms that are important precisely at low curvature. The simplest possibility is to consider a $\frac{1}{R}$ term in the Einstein-Hilbert action. Carrol et al also suggested that such a theory maybe suitable to derive cosmological models with late accelerating phase. Although the the theory with with the $\frac{1}{R}$ term in the Einstein's gravity accounts satisfactorily the present acceleration of the Universe, it is realised that inclusion of such terms in the Einstein-Hilbert action leads to instabilities[105].

Subsequently it has been shown that further addition of an R^2 term or an $\ln R$ term to the Einstein's gravitational action leads to consistent modified theory of gravity which may pass satisfactorily solar system tests, and fee from instability problems. It is known that the modified gravity with a positive power of the scalar curvature, namely R^2, in the EH action admits early inflation. In fact, the model $f(R) = R + \alpha R^2$ leads to accelerated expansion of the Universe, and this was the first model of inflation proposed by Starobinsky in 1980.This model is well consistent with the temperature anisotropies observed in CMB and thus it can be a viable alternative to the scalar field models of inflation. Reheating after inflation proceeds by a gravitational particle production during the oscillating phase of the Ricci scalar[1],[41],[105],[111]. The modified gravity with negative powers of the curvature in the Einstein-Hilbert action is recently becoming popular as it might effectively behave as a dark energy candidate. Consequently one such theory maybe able to describe the recent cosmic acceleration. Therefore it is reasonable to explore a theory which could accommodate an inflationary scenario at the early universe and an accelerating phase of expansion at late time followed by a matter dominated phase. As a result, the modified theory of gravity which contains both positive and negative powers of the Ricci Scalar R, with the general form of f(R) given by[105]:

$$f(R) = R + \alpha R^m + \beta \frac{1}{R^n} \tag{166}$$

where α, β to represent coupling constants, with arbitrary constants m and n are considered for exploring cosmological models. It is known that the R^m term dominates and it permits power law inflation if $1 \leq m \leq 2$, in the large curvature limit.

For m=2 and $\beta = 0$ we get inflation as we have seen above[105]. Recently, in the law curvature limit, a number of f(R) models have been proposed in order to accommodate the universe with late acceleration using a modified gravity, namely, $f(R) = R - \frac{\lambda}{R^n}$, with $n > 0$. In the metric approach, it was shown that the model is not suitable because it didn't permit a matter era. Recently it was also shown that the model $f(R) = R + \alpha R^m$ is not cosmologically viable because it does not permit a consistent scenario accommodating a matter dominated era at late time with $\alpha \sim t^{\frac{2}{3}}$, but instead it permits a radiation dominated era with $\alpha \sim t^{\frac{1}{2}}$. On the other hand the R^m model does permit a matter dominated universe but it fails to connect to the late accelerating phase[105].

It was shown(see[105]) that the models of the type where Lagrangian density, $f(R) = R - \frac{\lambda}{R^n}$ with n ¿ 0 and $f(R) = \alpha R^m$ with $m \neq 1$ are not viable for a realistic cosmological scenario as they do not permit matter epoch although late acceleration can be realized. Recently, modified gravity with power law in R, i.e., f(R)-gravity is examined and found that a large class of models including R^m model does not permit matter dominated universe. Tsujikawa (see[105]) derived observational signature of f(R) dark energy models that satisfy cosmological and local gravity constraints fairly well. The modified f(R)-gravity is found to be consistent with realistic cosmology in some cases. However, no definite physical criteria known so far to select a particular kind of theory capable of matching the data at all scales. However, modified gravity namely, $f(R) \sim e^R$ or $\log R$ may be useful to build a viable cosmological model as they permit a matter dominated phase before an accelerating phase of expansion[105].

5.3 f(R) in The Metric Formalism

Beginning from the Einstein-Hilbert modified action Eq.(165) and adding a matter term S_M, then the total action for F(R) gravity becomes[1],[41],[106],[107],[111],[112],[113],[114],[115]:

$$S_{met} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu},\psi) \tag{167}$$

where ψ denotes the matter fields. In the metric formalism we make a variation of the action with respect to the metric in order to obtain the field equations[1],[41],[106],[107],[111],[112],[113],[114],[115]:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla_\mu\nabla^\mu]f'(R) = 8\pi G T_{\mu\nu} \tag{168}$$

with the energy-momentum tensor[1],[41],[106],[107],[111],[112],[113],[114],[115]

to be given by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{169}$$

the prime denotes differentiation with respect to the argument, ∇_μ is the covariant derivative associated with the Levi-Civita connection, and $T_{\mu\nu}$ is the usual energy-momentum tensor.

It has to be stressed that there is a mathematical jump[41],[111] in deriving Eq.(168) from the action Eq.(167) having to do with the surface terms that appear in the variation: as in the case of the EinsteinHilbert action, the surface terms do not vanish just by fixing the metric on the boundary. For the EinsteinHilbert action, however, these terms gather into a total variation of a quantity. Therefore, it is possible to add a total divergence to the action in order to heal it and arrive to a well-defined variational principle (this is the well known GibbonsHawkingYork surface term (Gibbons and Hawking, 1977; York, 1972)). Unfortunately, the surface terms in the variation of the action Eq.(165) do not consist of a total variation of some quantity (the reader is urged to calculate the variation in order to verify this fact) and it is not possible to heal the action by just subtracting some surface term before performing the variation[41],[111].

The way out comes from the fact that the action includes higher order derivatives of the metric and, therefore, it should be possible to fix more degrees of freedom on the boundary than those of the metric itself[41],[111]. There is no unique prescription for such a fixing in the literature so far. Note also that the choice of fixing is not void of physical meaning, since it will be relevant for the Hamiltonian formulation of the theory. However, the field equations (168) would be unaffected by the fixing chosen and from a purely classical perspective, such as the one followed here, the field equations are all that one needs for a more detailed discussion on these issues]. Setting aside the complications of the variation we can now focus on the field equations (168). These are obviously fourth order partial differential equations in the metric, since R already includes second derivatives of the latter. For an action which is linear in R, the fourth order terms the last two on the left hand side vanish and the theory reduces to GR. Taking the trace of Eq.(168) leads to[41],[106],[111]:

$$f'(R)R - 2f(R) + 3\nabla_\mu \nabla^\mu f'(R) = 8\pi GT \tag{170}$$

where $T = g_{\mu\nu}T^{\mu\nu}$ relates R with T differentially and not algebraically as in GR, where $R = -8\pi GT$. This is already an indication that the field equations of f(R) theories will admit a larger variety of solutions than Einsteins theory. As an example, we mention here that the Jebsen- Birkhoffs theorem[111], stating that the Schwarzschild solution is the unique spherically symmetric vacuum solution, no longer holds in metric f(R) gravity. Without going into details, let us stress that T = 0 no longer implies that R = 0, or is even constant.

Eq.(170) will turn out to be very useful in studying various aspects of f(R) gravity, notably its stability and weak-field limit. For the moment, let us use it to make some remarks about maximally symmetric solutions. Recall that maximally symmetric solutions lead to a constant Ricci scalar. For R = constant and $T_{\mu\nu} = 0$, Eq.(170) reduces to[41],[111]:

$$f'R - 2f(R) = 0 \tag{171}$$

which, for a given f, is an algebraic equation in R. If R = 0 is a root of this equation and one takes this root, then Eq.(168) reduces to $R_{\mu\nu} = 0$ and the maximally symmetric solution is Minkowski space-time. On the other hand, if the root of Eq.(171) is R = C, where C is a constant, then Eq.(168) reduces to $R_{\mu\nu} = g_{\mu\nu}C/4$ and the maximally symmetric solution is de Sitter or anti-de Sitter space depending on the sign of C, just as in GR with a cosmological constant.

Another issue that should be stressed is that of energy conservation. In metric f(R) gravity the matter is minimally coupled to the metric. One can, therefore, use the usual arguments based on the invariance of the action under diffeomorphisms of the space-time manifold [coordinate transformations followed by a pullback, with the field $x^\mu \to x^\mu + \xi^\mu$ vanishing on the boundary of the space-time region considered, leave the physics unchanged, see (Wald, 1984)] to show that $T_{\mu\nu}$ is divergence-free. The same can be done at the level of the field equations: a brute force calculation reveals that the left hand side of Eq.(168) is divergence-free (generalized Bianchi identity) implying that $\nabla_\mu T^{\mu\nu} = 0$ (Koivisto,2006)[41],[111],[112],[115].

Finally, let us note that it is possible to write the field equations in the form of Einstein equations with an effective stress-energy tensor composed of curvature terms moved to the right hand side[41],[111]. This approach is questionable in principle (the theory is not Einsteins theory and it is artificial to force upon it an interpretation in terms of Einstein equations) but, in practice, it has been proved to be useful in scalar-tensor gravity. Specifically, Eq.(168) can be written[111] as:

$$G_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{f'(R)} + g_{\mu\nu}\frac{[f(R) - Rf'(R)]}{2f'(R)} + \frac{[\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\nabla_\mu\nabla^\mu f'(R)]}{f'(R)} \tag{172}$$

or

$$G_{\mu\nu} = \frac{8\pi G}{f'(R)}(T_{\mu\nu} + T^{(eff)}_{\mu\nu}) \tag{173}$$

with

$$T^{(eff)}_{\mu\nu} = \frac{1}{8\pi G}[g_{\mu\nu}\frac{f(R) - Rf'(R)}{2} + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\nabla_\mu\nabla^\mu f'(R)] \tag{174}$$

here the quantity $G_{eff} = \frac{G}{f'(R)}$can be regarded as the effective gravitational coupling strength in analogy to what is done in scalar-tensor gravity and positivity of G_{eff} imposes that $f'(R) > 0$. Furthermore $T^{(eff)}_{\mu\nu}$ is an effective stress-energy tensor which does not have the canonical form quadratic in the first derivatives of the field $f'(R)$, but contains terms linear in the second derivatives. The effective energy density derived from it is not positive-definite and none of the energy conditions holds[41],[111].

5.4 f(R) in The Palatini Formalism

We have already mentioned that the Einstein equations can be derived using, instead of the standard metric variation of the EinsteinHilbert action, the Palatini formalism, i.e., an independent variation with respect to the metric and an independent connection (Palatini variation).The action is formally the same but now the Riemmann tensor and the Ricci tensor are constructed with the independent connection[1],[41],[111]. Note that the metric is not needed to obtain the latter from the former. For clarity of notation, we denote the Ricci tensor constructed with this independent connection as $R_{\mu\nu}$ and the corresponding Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$. The action now takes the form[1],[41],[111],[112],[113],[114],[119]:

$$S_{pal} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu}, \psi) \tag{175}$$

We note that the matter action SM is assumed to depend only on the metric and the matter fields and not on the independent connection. This assumption is crucial for the derivation of Einsteins equations from the linear version of the action (175) and is the main feature of the Palatini formalism.

It has already been mentioned that this assumption has consequences for the physical meaning of the independent connection[1],[41],[111],[119]. Let us elaborate on this: recall that an affine connection usually defines parallel transport and the covariant derivative. On the other hand, the matter action S_M is supposed to be a generally covariant scalar which includes derivatives of the matter fields. Therefore, these derivatives ought to be covariant derivatives for a general matter field. Exceptions exist, such as a scalar field, for which a covariant and a partial derivative coincide, and the electromagnetic field, for which one can write a covariant action without the use of the covariant derivative. However, S_M should include all possible fields. Therefore, assuming that S_M is independent of the connection can imply one of two things[41],[111]: either we are restricting ourselves to specific fields, or we are implicitly assuming that it is the Levi-Civita connection of the metric that actually defines parallel transport. Since the first option is implausibly limiting for a gravitational theory, we are left with the conclusion that the independent connection $\Gamma^{\lambda}{}_{\mu\nu}$ does not define parallel transport or the covariant derivative and the geometry is actually pseudo- Riemannian. The covariant derivative is actually defined by the Levi-Civita connection of the metric $\{^{\mu}_{\alpha\beta}\}$.

This also implies that Palatini f(R) gravity is a metric theory[1],[41],[111] in the sense that it satisfies the metric postulates (Will, 1981). Matter is minimally coupled to the metric and not coupled to any other fields. Once again, as in GR or metric f(R) gravity, one could use diffeomorphism invariance to show that the stress energy tensor is conserved by the covariant derivative defined with the Levi-Civita connection of the metric, i.e., $\nabla_\mu T^{\mu\nu} = 0$ ($\hat{\nabla}_\mu T^{\mu\nu} \neq 0$). This can also be shown by using the field equations, which we will present shortly, in order to calculate the divergence of $T_{\mu\nu}$ with respect to the Levi-Civita connection of the metric and show that it vanishes. Clearly then, Palatini f(R) gravity is a metric theory according to the definition (not to be confused with the term metric in metric f(R) gravity, which simply refers to the fact that one only varies the action with respect to the metric). Conventionally thinking, as a consequence of the covariant conservation of the matter energy-momentum tensor, test particles should follow geodesics of the metric in Palatini f(R) gravity. This can be seen by considering a dust fluid with $T_{\mu\nu} = \rho u_\mu u_\nu$ and projecting the conservation equation $\nabla^\alpha T_{\alpha\beta} = 0$ onto the fluid four-velocity u^α.Similarly, theories that satisfy the metric postulates are supposed to satisfy the Einstein Equivalence Principle as well(Will, 1981).Varying the action (175) independently with respect to the metric and the connection, respectively, and using the formula[41],[111],[119]:

$$\delta R_{\mu\nu} = \hat{\nabla}_\lambda \delta\Gamma^\lambda{}_{\mu\nu} - \hat{\nabla}_\nu - \delta\Gamma^\lambda{}_{\mu\lambda} \tag{176}$$

gives

$$f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{177}$$

$$-\hat{\nabla}_\lambda(\sqrt{-g}f'(R)g^{\mu\nu}) + \hat{\nabla}_\sigma(\sqrt{-g}f'(R)g^{\sigma(\mu})\delta^{\nu)}_\lambda = 0 \tag{178}$$

where $T_{\mu\nu}$ is the usual energy momentum tensor,$\hat{\nabla}_\mu$ denotes the covariant derivative defined with the independent connection $\Gamma^\lambda{}_{\mu\nu}$, and $(\mu\nu), [\mu\nu]$ denotes symmetrisation or anti-symmetrisation over the indices μ and ν respectively. Taking now the trace of Eq.(178) we get[41],[111],[112],[113],[114],[119]:

$$\hat{\nabla}_\alpha(\sqrt{-g}f'(R)g^{\alpha\mu}) = 0 \tag{179}$$

Hence the field equations can be written as[41],[111],[112],[113],[114],[119]:

$$f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{180}$$

$$\hat{\nabla}_\alpha(\sqrt{-g}f'(R)g^{\beta\mu}) = 0 \qquad (181)$$

It is now easy to see how the Palatini formalism leads to GR when f(R)=R. In this case $f'(R) = 1$ and Eq.(181) becomes the definition of the Levi-Civita connection for the initially independent connection ${\Gamma^\lambda}_{\mu\nu}$. Then, $R_{\mu\nu}$ is the usual Ricci tensor and R the Ricci scalar and Eq.(180) gives the Einstein field equations. This reproduces the result that can be found in textbooks (Misner et al., 1973; Wald, 1984). Note that in the Palatini formalism for GR, the fact that the connection turns out to be the Levi-Civita one is a dynamical feature instead of an a priori assumption. It is now evident that generalizing the action to be a general function of R in the Palatini formalism is just as natural as it is to generalize the EinsteinHilbert action in the metric formalism. Remarkably, even though the two formalisms give the same results for linear actions, they lead to different results for more general actions[1],[41],[111],[112],[113],[114],[119].

Taking the trace of Eq.(180) we get[41],[111],[114],[119]:

$$f'(R)R - 2f(R) = 8\pi GT \qquad (182)$$

For all cases in which T = 0, including vacuum and electro-vacuum, R will therefore be a constant and a root of the equation:

$$f'(R)R - 2f(R) = 0 \qquad (183)$$

Eq.(183) can also be identically satisfied if $f(R) \propto R^2$. This very particular choice for f leads to a conformally invariant theory (Ferraris et al., 1992). As is apparent from Eq.(182),if $f(R) \propto R^2$ then only conformally invariant matter, for which T = 0 identically, can be coupled to gravity. Matter is not generically conformally invariant though, and so this particular choice of f is not suitable for a low energy theory of gravity[41],[111].

Consider now Eq.(181) and we define a metric conformal to $g_{\mu\nu}$ such that[41],[111],[114],[119]:

$$h_{\mu\nu} = f'(R)g_{\mu\nu} \qquad (184)$$

and therefore:

$$\sqrt{-h}h^{\mu\nu} = f'(R)\sqrt{-g}g^{\mu\nu} \qquad (185)$$

Then Eq.(181)becomes the definition of the Levi-Civita connection[41],[111],[114],[119] of $h_{\mu\nu}$ and can be solved algebraically to give:

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} h^{\lambda\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) \tag{186}$$

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}\frac{g^{\lambda\sigma}}{f'(R)}[\partial_\mu(f'(R)g_{\nu\sigma}) + \partial_\nu(f'(R)g_{\mu\sigma}) - \partial_\sigma(f'(R)g_{\mu\nu})] \tag{187}$$

The Ricci tensor transforms under conformal transformations[41],[111],[114],[119] as:

$$R_{\mu\nu} \to R_{\mu\nu} + \frac{3}{2}\frac{1}{(f'(R))^2}(\nabla_\mu f'(R))(\nabla_\nu f'(R)) - \frac{1}{f'(R)}(\nabla_\mu\nabla_\nu - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\nabla_\alpha)f'(R) \tag{188}$$

contracting with $g_{\mu\nu}$ we then get:

$$R \to R + \frac{3}{2}\frac{1}{(f'(R))^2}(\nabla_\mu f'(R))(\nabla^\mu f'(R)) + \frac{3}{f'(R)}\nabla_\mu\nabla^\mu f'(R) \tag{189}$$

Plugging Eq.(188) and Eq.(189) into Eq.(180) we get:

$$G_{\mu\nu} = \frac{8\pi G}{f'}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - \frac{f}{f'}) + \frac{1}{f'}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\alpha\nabla_\alpha)f' - \frac{3}{2f'^2}[(\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2}g_{\mu\nu}(\nabla f')^2] \tag{190}$$

Notice that, assuming that we know the root of Eq.(182),R = R(T), we have completely eliminated the independent connection from this equation. Therefore, we have successfully reduced the number of field equations to one and at the same time both sides of Eq.(190) depend only on the metric and the matter fields. In a sense, the theory has been brought to the form of GR with a modified source.We can now deduce the following[41],[111],[114],[119]:

• If f(R)=R then the theory reduces to General Relativity

• For matter fields with T =0, due to Eq.(183),R and consequently f(R) and $f'(R)$ are constants and the theory reduces to GR with a cosmological constant and a modified coupling constant G/f'. If we denote the value of R when T = 0 as R_0, then the value of the cosmological constant is given by:

$$\frac{1}{2}(R_0 - \frac{f(R_0)}{f'(R_0)}) = \frac{R_0}{4} \tag{191}$$

• In the general case $T \neq 0$, the modified source on the right hand side of Eq.(190) includes derivatives of the stress-energy tensor, unlike in GR. These are implicit in the last two terms of Eq.(190), since f' is in practice a function of T, given that $f' = f'(R)$ and R = R(T).

5.5 f(R) in The Metric-Affine Formalism

As we already pointed out, the Palatini formalism of f(R) gravity relies on the crucial assumption that the matter action does not depend on the independent connection[1],[41],[111]. We also argued that this assumption relegates this connection to the role of some sort of auxiliary field and the connection[41],[111]. carrying the usual geometrical meaning of the parallel transport and definition of the covariant derivative remains the Levi-Civita connection of the metric. As we have already discussed in Subsection3.2, in metric-affine theories the metric and the connection are independent, as in the case of the Palatini Formalism, however in this metric-affine theories of gravity there is a direct coupling between matter and connection[1],[41],[111], as the action includes covariant derivative of matter fields with the covariant derivatives defined using the connection. Hence the matter action would be a function of the independent metric and connection and the matter fields $S_M = S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi)$. We have seen the metric-affine action for gravity in Subsection3.2, hence replacing R with f(R) the action is therefore[1],[41],[111],[112],[113],[117],[118]:

$$S_{ma} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi) \tag{192}$$

In Subsection3.2 we discussed in detail what is the general set-up for metric-affine theories of gravity. We know that in such a theory the metric and connection are independent, however, some part of the connection is still related to the metric, as the non-metricity tensor is set to be equal to zero. In this case the connection is left to be completely unconstrained and hence to be determined by the field equations[41],[111].But leaving the connection completely unconstrained comes with some problems.Lets consider the projective transformation[41],[111]:

$$\Gamma^{\lambda}{}_{\mu\nu} \rightarrow \Gamma^{\lambda}{}_{\mu\nu} + \delta^{\lambda}{}_{\mu}\xi_{\nu} \tag{193}$$

where ξ_{ν} is an arbitrary covariant vector field.It can be shown that the Ricci tensor transforms as[41],[111],[117],[118]:

$$R_{\mu\nu} \rightarrow R_{\mu\nu} - 2\partial_{[\mu}\xi_{\nu]} \tag{194}$$

However given that the metric is symmetric, this implies that the curvature scalar does not change and hence[41],[111],[117],[118]:

$$R \rightarrow R \tag{195}$$

or in other words R is invariant under projective transformations. Therefore the Einstein-Hilbert action or any other action that is a function of R, as in our case, is projective invariant in metric-affine gravity However, the matter action is not generically projective invariant and this would be the cause of an inconsistency in the field equations.We could try to avoid this problem by generalizing the gravitational action in order to break projective invariance. This can be done in several ways, such as allowing for the metric to be non-symmetric as well, adding higher order curvature invariants or terms including the Cartan torsion tensor.However, if one wants to stay within the framework of f(R) gravity, which is our subject here, then there is only one way to cure this problem: to somehow constrain the connection. In fact, it is evident from Eq.(193) that, if the connection were symmetric, projective invariance would be broken[41],[111],[117],[118].

Lets see again what is the meaning of projective invariance, which is very similar to gauge invariance in the the theory of Electromagnetism. Projective invariance tells us that the corresponding field can be determined up to a projective transformation Eq.(193). In this case the field is just the connections.If we want to break this invariance we have to fix some degrees of freedom in the same way we do with gauge fixing.The number of degrees of freedom which we need to fix is obviously the number of the components of the four-vector used,for the transformation, i.e., simply four.This tells us that the connection satisfies some constraints but it cannot be assumed to be the most general connection we can construct[41],[111].

The degrees of freedom we need to fix is four and are related to the non-symmetric part of the connection. Then we should set[41],[111],[117],[118]:

$$S = S_{\mu} = S_{\sigma\mu}{}^{\sigma} = 0 \tag{196}$$

without, of course, this to mean that the part of the connection $\Gamma_{\mu} = S_{\sigma\mu}{}^{\sigma}$ must vanish, but mainly that $\Gamma_{\mu} = S_{\sigma\mu}{}^{\sigma} = \Gamma_{\mu} = S_{\mu\sigma}{}^{\sigma}$ This constrained can be imposed by simply adding a Lagrange multiplier B^{μ}. Hence the action for the most general metric-affine f(R) theory of gravity is given by:

$$S_{ma} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi) + S_{LM} \tag{197}$$

where S_{LM} is the action of the Lagrange Multiplier given by[41],[111],[117],[118]:

$$S_{LM} = \int d^4x\sqrt{-g}B^{\mu}S_{\mu} \tag{198}$$

We now vary the action independently with respect to the metric, connection, and lagrange Multiplier in order to get the field equations[41],[111],[112],[113],[117],[118]:

$$f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{199}$$

$$\frac{1}{\sqrt{-g}}[-\hat{\nabla}_{\lambda}(\sqrt{-g}f'(R)g^{\mu\nu}) + \hat{\nabla}_{\sigma}(\sqrt{-g}f'(R)g^{\mu\sigma})\delta^{\nu}{}_{\lambda}] + 2f'(R)(g^{\mu\nu}S_{\lambda\sigma}{}^{\sigma} - g^{\mu\rho}S_{\rho\sigma}{}^{\sigma}\delta^{\nu}{}_{\lambda}$$

$$+g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu}) = 8\pi G(\Delta_{\lambda}{}^{\mu\nu} - B^{[\mu}\delta^{\nu]}{}_{\lambda}) \tag{200}$$

$$S_{\lambda\sigma}{}^{\sigma} = 0 \tag{201}$$

Where $\Delta_{\lambda}{}^{\mu\nu}$ is called the hyper-momentum tensor and mimics the role of the energy-momentum tensor. Furthermore $S_{\sigma\lambda}{}^{\nu}$ is the Cartan-Torsion tensor as discussed in Subsection3.2 and is equal to the anti-symmetric part of the connection. Taking the trace of Eq.(199) over the indices μ and ν and using Eq.(200) we have that[41],[111],[112],[113],[117],[118]:

$$B^{\mu} = \frac{2}{3}\Delta_{\sigma}{}^{\sigma\mu} \tag{202}$$

Hence we can now write the field equations as[41],[111],[112],[113],[117],[118]:

$$f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{203}$$

$$\frac{1}{\sqrt{-g}}[-\hat{\nabla}_\lambda(\sqrt{-g}f'(R)g^{\mu\nu}) + \hat{\nabla}_\sigma(\sqrt{-g}f'(R)g^{\mu\sigma})\delta^\nu{}_\lambda]$$

$$+2f'(R)g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu} = 8\pi G(\Delta_\lambda{}^{\mu\nu} - \frac{2}{3}\Delta_\sigma{}^{\sigma[\nu}\delta^{\nu]}_\lambda) \tag{204}$$

$$S_{\lambda\sigma}{}^{\sigma} = 0 \tag{205}$$

By splitting Eq.(204) into a symmetric and an antisymmetric part and performing the appropriate contractions and manipulations we find that[41],[111],[117],[118]:

$$\Delta_\lambda{}^{[\mu\nu]} = 0 \Rightarrow S_{\mu\nu}{}^{\lambda} = 0 \tag{206}$$

Hence this has the two following implications[41],[111],[117],[118]:

• Any torsion is introduced by matter fields for which $\Delta_\lambda{}^{[\mu\nu]}$ is non-vanishing.
• Torsion is not propagating, since it is given algebraically in terms of the matter fields through $\Delta_\lambda{}^{[\mu\nu]}$ It can, therefore, only be detected in the presence of such matter fields. In the absence of the latter, space-time will have no torsion.

Similarly we can use the symmetrized version of Eq.(204) to show that the symmetric part of the hyper-momentum $\Delta_\lambda{}^{(\mu\nu)}$ is algebraically related to the non-metricity $Q_{\mu\nu\lambda}$. Therefore, matter fields with non-vanishing $\Delta_\lambda{}^{(\mu\nu)}$ will introduce non-metricity.In this case things are slightly more complicated because part of the non-metricity is also due to the functional form of the Lagrangian itself. There are, however, certain types of matter fields for which $\Delta_\lambda{}^{\mu\nu} = 0$ such as[41],[111],[117,[118]:

•A scalar field, since in this case the covariant derivative can be replaced with a partial derivative.Therefore, the connection does not enter the matter action.
• The electromagnetic field (and gauge fields in general), since the electromagnetic field tensor $F_{\mu\nu}$ is defined in a covariant manner using the exterior derivative. This definition remains unaffected when torsion is included.

On the contrary, particles with spin, such as Dirac fields, generically have a non-vanishing hyper-momentum and will, therefore, introduce torsion. A more complicated case is that of a perfect fluid with vanishing vorticity. If we set torsion aside, or if we consider a fluid describing particles that would initially not introduce any torsion then, as for a usual perfect fluid in GR, the matter action can be written in terms of three scalars: the energy density, the pressure, and the velocity potential. Therefore such a fluid will lead to a vanishing ${\Delta_\lambda}^{\mu\nu}$.However, complications arise when torsion is taken into account: Even though it can be argued that the spins of the individual particles composing the fluids will be randomly oriented, and therefore the expectation value for the spin should add up to zero, fluctuations around this value will affect space-time. Of course, such effects will be largely suppressed, especially in situations in which the energy density is small, such as late time cosmology[41],[111],[117],[118].

Because of Eq.(206) we can now see that the field equations of metric-affine f(R) gravity reduce to the field equations (177) and (178) of the Palatini f(R) gravity in the case where ${\Delta_\lambda}^{\mu\nu} = 0$. Furthermore, in vacuum where the energy-momentum tensor $T_{\mu\nu}$ vanishes, the field equations of the Palatini f(R) gravity reduce to the Einstein Field equations with an effective cosmological constant[41],[111].

In conclusion, metric-affine f(R) gravity appears to be the most general case of f(R) gravity. It includes enriched phenomenology, such as matter-induced non-metricity and torsion. It is worth stressing that torsion comes quite naturally, since it is actually introduced by particles with spin (excluding gauge fields). The theory reduces to GR in vacuum or for conformally invariant types of matter, such as the electromagnetic field, and departs from GR in the same way that Palatini f(R) gravity does for most matter fields that are usually studied as sources of gravity.However, at the same time, it exhibits new phenomenology in less studied cases, such as in the presence of Dirac fields, which include torsion and non-metricity. Finally let us stretch that the Palatini f(R) gravity is really a metric theory in contrast with the metric-affine f(R) gravity which is not a metric theory. Therefore $T^{\mu\nu}$ is not divergent free with respect to the covariant derivative defined with the Levi-Civita connection, in the metric-affine f(R) theory. Furthermore the physical meaning of the above statement is subtle as in metric-affine gravity $T^{\mu\nu}$ does not have the usual meaning of the energy-momentum tensor as in GR, for instance, it does not reduce to the special relativistic tensor at an appropriate limit and at the same time there is also another quantity, the hyper-momentum, which describes matter characteristics[41],[111].

6 Summary-Conclusions

In this paper we have presented a detailed review of Einstein's theory of General Relativity which is the most successful theory of Gravity. We have reviewed the main principles that underlie this theory, the Einstein-Hilbert action from which we can derive the Einstein Field equations that describe the theory of General Relativity, and we have argued that GR is a classical theory that respects the principle of covariance, exhibits universal coupling to all matter fields, and satisfies the Einstein Field equations, a set of ten(10) non-linear PDE's. Hence if we deviate from the above axioms of GR then we have what we call modified gravity. We also presented the various astrophysical tests, such as the perihelion precession of mercury's orbit, the deflection of light by the sun, gravitational red-shift and others, that are used to test and actually verify the validity of the theory.

In Chapter3 we have examined different formulations of GR, such as the Palatini formalism, the Metric-Affine gravity, the Vierbein formalism, and briefly discussed a few others. We have seen that in order to derive the Einstein Field equations we make the assumptions of Riemannian geometry, the vanishing of torsion, and we also assume that the connection is metric compatible. Making a variation of the action with respect to the metric we obtain the Einstein Field equations. In the Palatini formalism, however, we don't immediately assume that the connection is metric compatible. As the matter of fact we assume that the connection and the metric are independent, but the matter is still coupled to the metric, and that R is a function of the connection and not the usual curvature scalar. Varying the action with respect to the metric we get that the connection is indeed the Levi-Civita connection, and varying the Einstein-Hilbert action with respect to the connection we get the Einstein's field equations. The Palatini procedure when dealing with the Einstein-Hilbert action is that we derive the compatibility os the connection with the metric rather than assume this.
One further procedure is to keep the metric and connection independent and this time allow matter to couple not only to the metric but also to the connection. We still assume exactly what we have assumed in the Palatini formalism, however, in this case there is an additional condition: Matter can couple to the connection. This is the Metric-Affine formalism. At the end we can derive the vanishing of torsion and non-metricity from the Einstein-Hilbert action and of course recover the Einstein field equations.

In general what we have seen so far is that if we consider the Einstein-Hilbert and use the metric formalism, or the Palatini and metric-affine formalisms, we get the same field equations, namely, Einstein's field equations. However, for alternative theories of gravity other than General Relativity, different procedures and formalisms, given different field equations.

We have also presented in Chapter4 a possible modification of GR, namely, Chern-Simons modified theory, that is second order in curvature and therefore plausible from a high energy perspective[33],[59]. We have shown that the CS correction arises naturally in the Standard Model as a gravitational (ABJ) anomaly-cancellation mechanism, and it further required in String Theory to cancel the Green-Schwarz anomaly[33],[59]. We have also shown that CS gravity has a preferred direction with respect to the gravity waves that are produced during inflation, hence the chiral anomaly works together with inflation to amplify the production of leptons, leading to a viable model of leptogenesis[33],[59],[68],[69]. We have seen that the CS modified gravity yields the same physics as does the Classical GR. For example, the Schwarzchild solution holds without any modification because the CS correction vanish for this space-time and hence the modified theory passes the three classical tests of GR: 1) Perihelion advance of Mercury 2) The bending of light by the Sun 3) The slowing down of of clocks by gravity. Finally if the CS corrections are truly present in nature then they would be quantum suppressed and hence we need to examine in more detail the coupling strength in front of the CS correction that would be quantum suppressed at least at the electro-weak level, or even at Planck level[33].

In Chapter5 we have presented the very-well known modifications of GR, called f(R) theories of gravity as the gravitational part of the action is a function of the Ricci curvature scalar. This could be a linear function, or non-linear. There are several models examined in many papers, such as [111], where the reader can have a detailed review and analysis. We have presented some toy models, for example the R^2 model admits early inflation, other models may admit early inflation or late acceleration or even both, but it depends on the model and the assumptions made. We have discussed f(R) theories of gravity in the metric formalism, the Palatini Formalism and the metric-affine formalism. The corresponding field equations depend on whether the theory is linear or not, in R. For example if f(R)=R in the Palatini formalism then this reduces identically to GR. It turns out that the metric-affine f(R) gravity is the most general case of f(R) theories. Matter induces torsion and non-metricity and hence an enriched phenomenology. In vacuum it reduces to GR, and deviates from GR the same way the Palatini f(R) gravity does for most matter fields that are considered to be sources of gravity. We have to emphasize the fact that the Palatini f(R) gravity is a metric theory, whether the metric-affine f(R) gravity is not.

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